Battle Sheep is PSPACE-complete

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Talk Plan

- Describe Battle Sheep
- Play Battle Sheep
- Plug Sprouts 2026
- PSPACE reduction overview
- Show some gadgets

Note: interruptible!

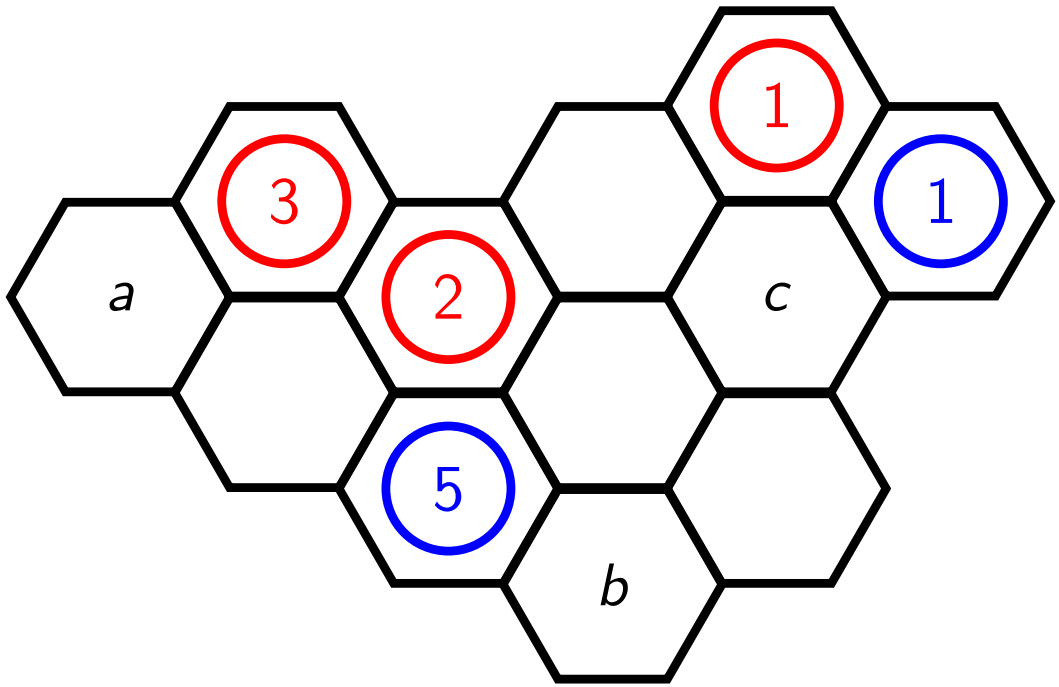
Battle Sheep

Actual published game!

Battle Sheep

- Start: sheep token stack along outer border.
- Game Turn:
  - Choose one of your stacks with more than one sheep. (Say k)
  - Choose some of those tokens (1 to $k-1$) to move as far as you can go in one direction.
  - Have to leave at least one token behind.
- If you can't move on your turn, you lose.
  - (Ignoring actual tiebreaking rule)

Battle Sheep



Battle Sheep

Let's play!
`https://kyleburke.info/DB/combGames/battleSheep.html`
(Recommendation: Width 6, Height 5.)

Battle Sheep

Battle Sheep is a Combinatorial Game!

- Two players ✓(actual game plays 2-4)
- Alternate Turns ✓
- No randomness ✓
- No hidden information ✓
- Normal Play: last person to play wins ✓

Use smallcaps: Battle Sheep

Sprouts 2026

We're playing Battle Sheep at Sprouts 2026

- Sprouts: Abstract games conference where undergrads present
- `https://kyleburke.info/sprouts/sprouts2026/`
- Sprouts 2026: April (Day TBD)
- Held online (Zoom)
- Fun for everyone:
  - Game tournament! 2026 Battle Sheep
  - Computer Tournament (JavaScript)
  - Lots of chatting.

PSPACE-completeness

- "Good" games have simple rules, but tough strategies.
- Not fun if players can always make the best move.
- Best Move $\approx$ Who can win
- Computational Complexity to the rescue!
  - Complexity Classes: categories of problems based on feasibility of running time.
  - P: Polymonial Time: problems that can be solved in time polynomial to the input size.
  - Problems in P are "tractible"; can be solved "efficiently".
  - $\mathrm{EXPTIME}$: problems that require exponential time to solve.
  - P vs NP: $\mathrm{P} \subseteq \mathrm{NP}$, but we don't know whether $\mathrm{P} = \mathrm{NP}$ or $\mathrm{P} \subsetneq \mathrm{NP}$.
  - Many people assume $\mathrm{P} \neq \mathrm{NP}$.

PSPACE-completeness

- PSPACE: computational problems solveable with polynomial-size memory/disk. (No time constraint.)
- $\mathrm{P} \subseteq \mathrm{NP} \subseteq \mathrm{PSPACE} \subseteq \mathrm{EXPTIME}$
- Hardest problems in PSPACE (-complete) have no known polynomial-time algorithm. Includes periodic scheduling, deadlock detection, and Amazons, Hex, Konane.
- In PSPACE: Battle Sheep $\in$ PSPACE ✓
- Need to show: PSPACE-hard: every problem in PSPACE can be transformed into an instance of Battle Sheep.
- Inclusion and Hardness: Battle Sheep is PSPACE-complete

PSPACE-completeness

- Prove hardness by finding a reduction.
- f : another PSPACE-hard problem $\rightarrow$ Battle Sheep
  - Preserve winnability: 1st player wins $x \Leftrightarrow$ 1st player wins $f(x)$
  - $f(x)$ computable in time polynomial in size of x
- Then, we can efficiently solve Battle Sheep $\Leftrightarrow$ we can efficiently solve all of PSPACE.
  - If `battleSheepSolver` solves it in polynomial time, then...
  - `function pspaceSolver(A):`

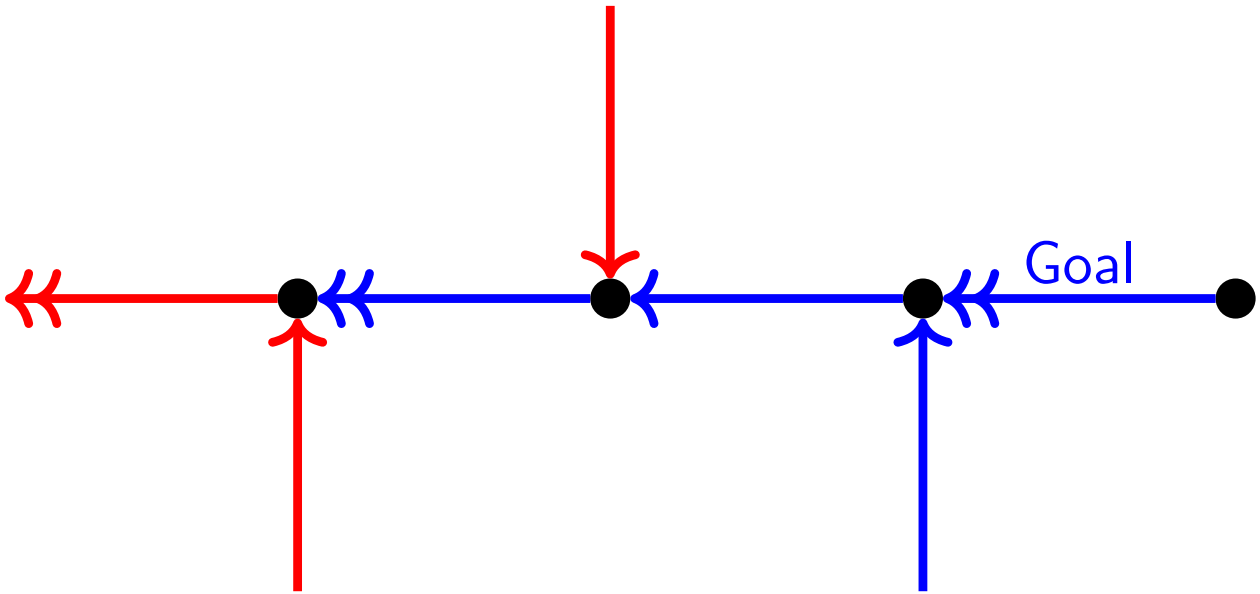
    ```
        bSheep = f(A)
        result = battleSheepSolver(bSheep)
        return result
    ```
  - Now `pspaceSolver` efficiently solves the original problem!
- A working reduction means Battle Sheep is PSPACE-hard.

B2CL

PSPACE-hard from Bounded Two-Player Constraint Logic (B2CL):

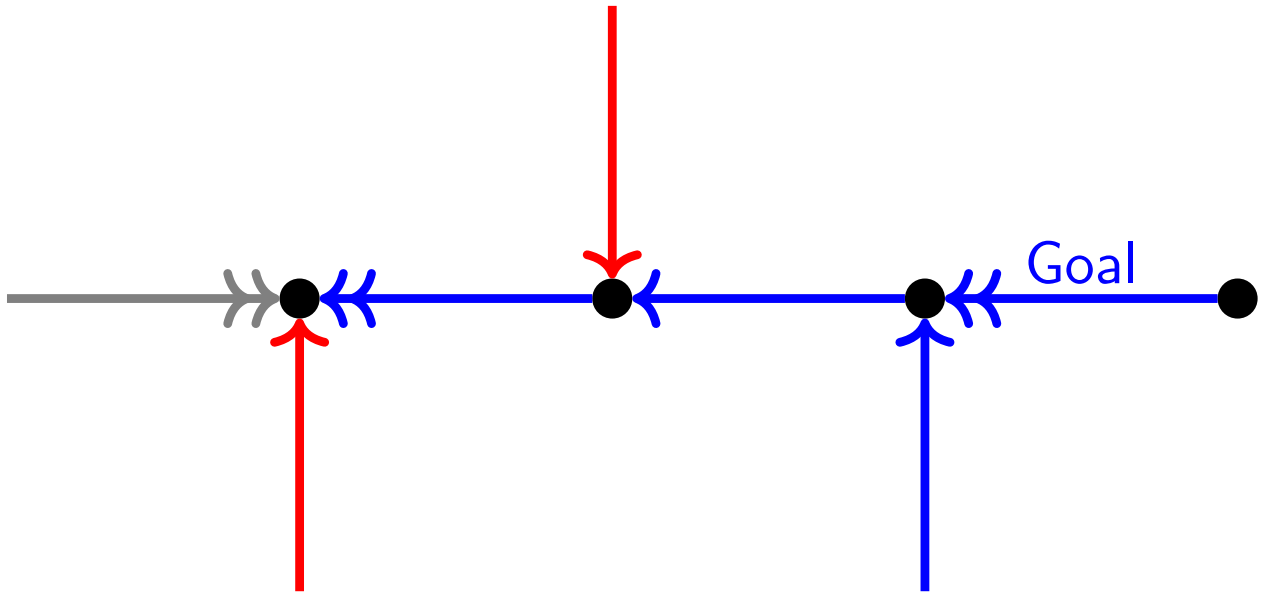
- B2CL is a game about flipping directed edges on graph.
- "Bounded": each edge can only be flipped once.
- "Two-Player": Blue and Red edges.
- Weight 1 and weight 2 edges.
- Each vertex must have in-weight of 2+.
- Win by flipping your goal edge.

B2CL



If Red flips left edge, Blue can win in three turns.

B2CL



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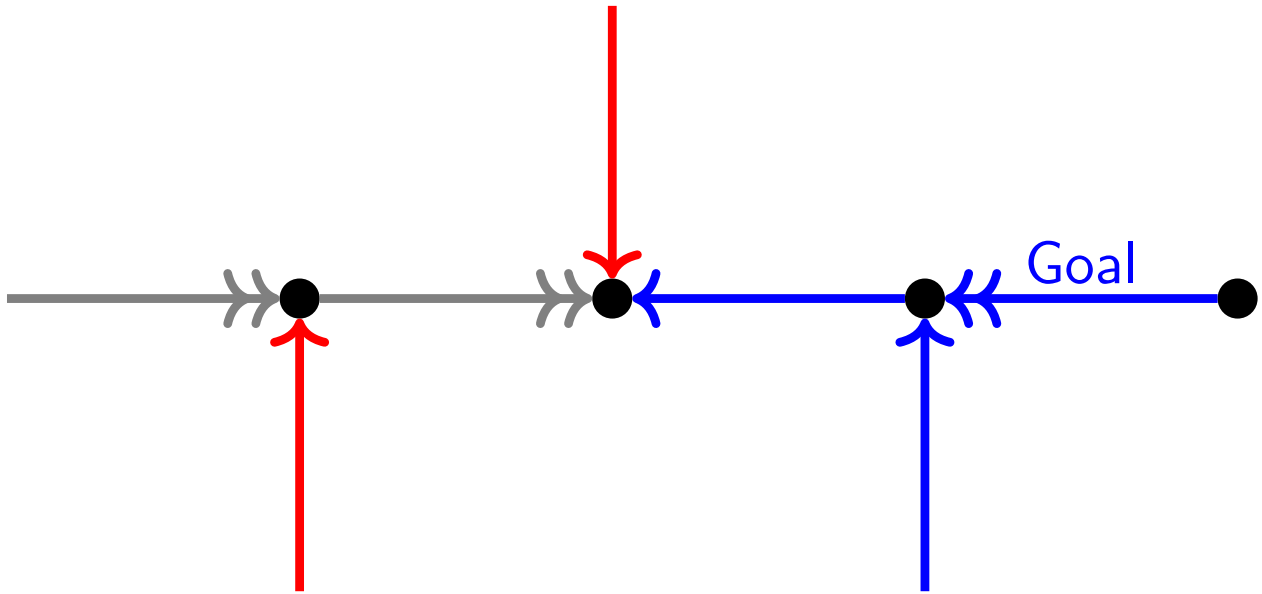
B2CL



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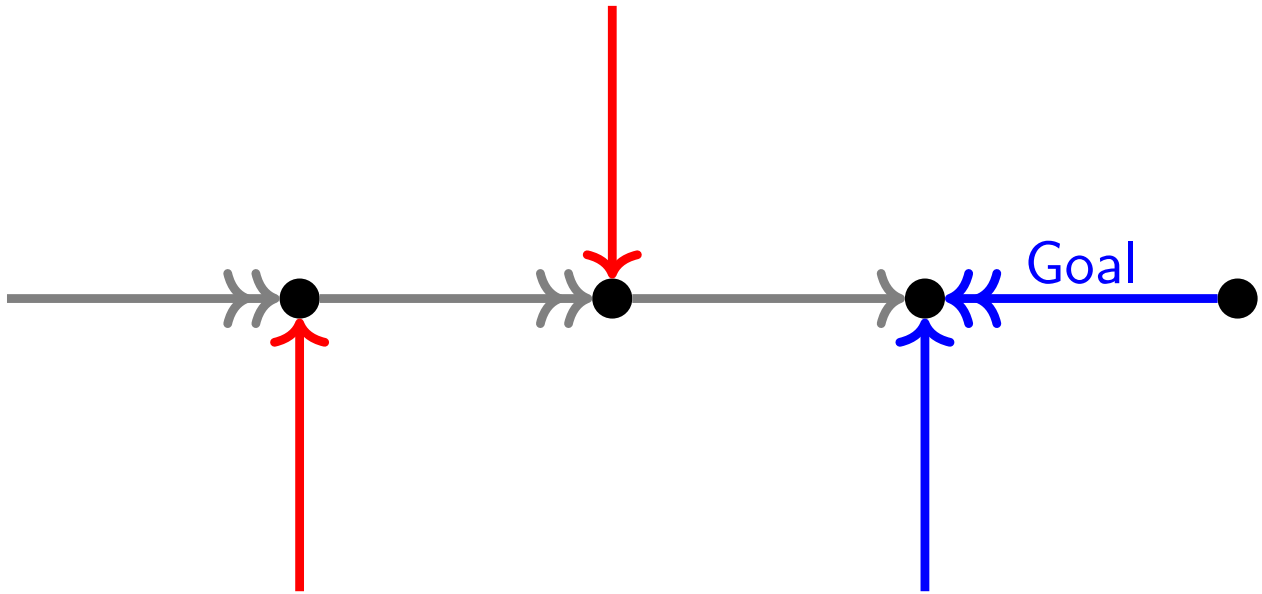
B2CL



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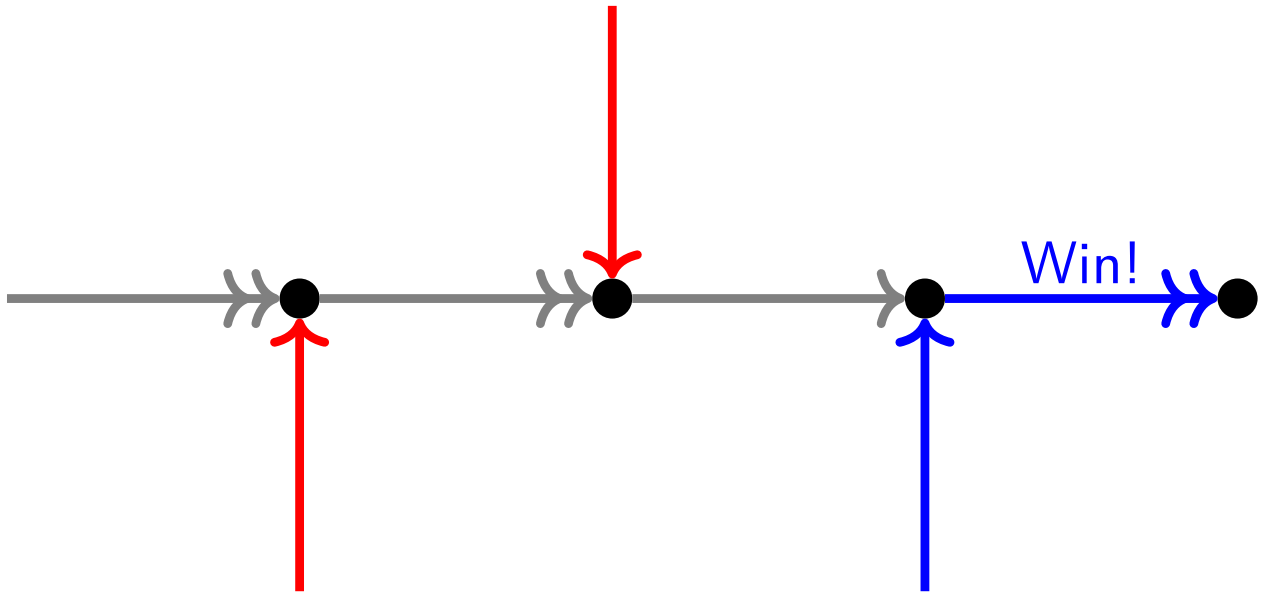
B2CL



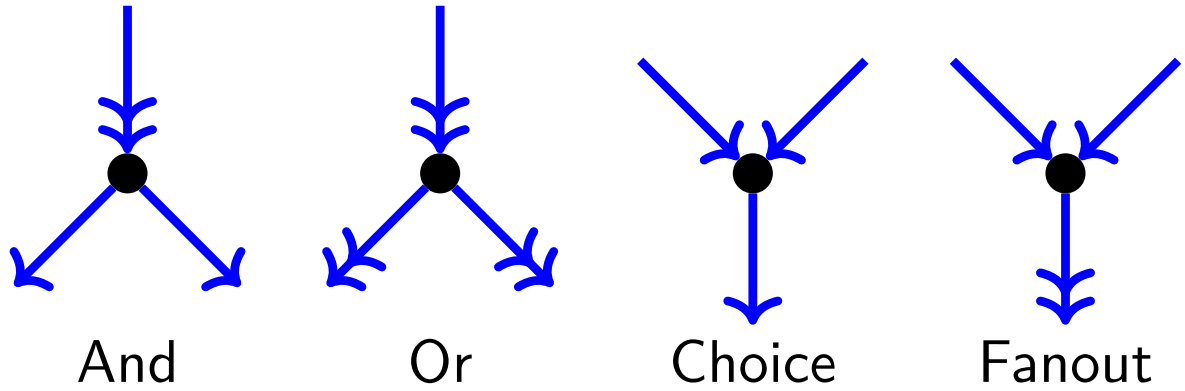
If Red flips left edge, Blue can win in three turns.

B2CL

Let's play a B2CL variant!
Note: no goals, normal play instead.
`https://kyleburke.info/DB/combGames/normalBoundedConstraintLogic.html`
(Recommend: 4 x 5)

B2CL

Amazingly, PSPACE-hard with only five vertex types:



Variable

Use "activate" term for positive/True signal.
(Via reduction from Positive CNF.)

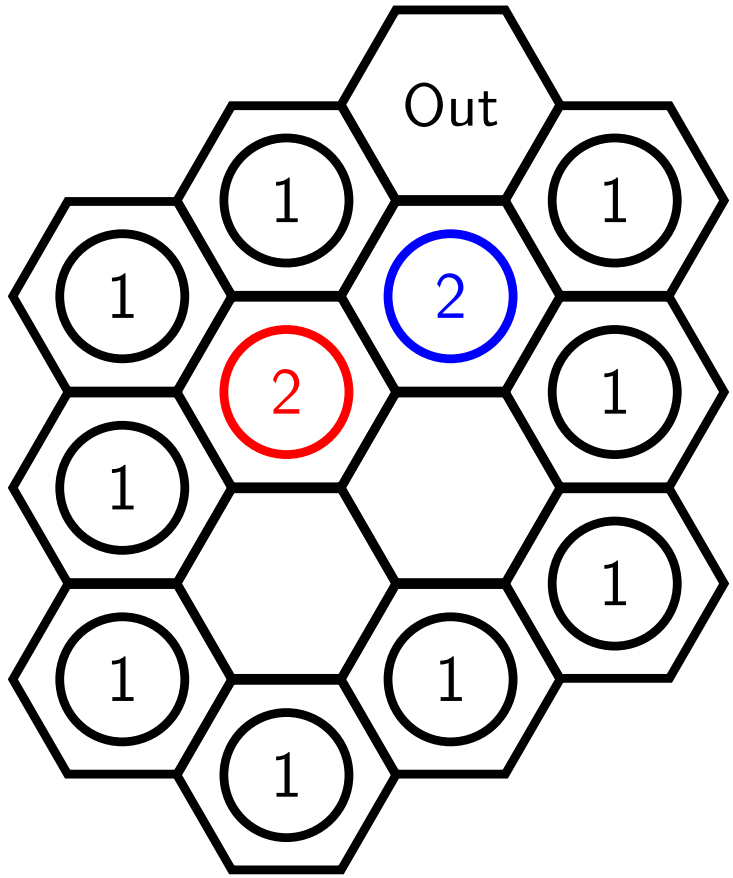
B2CL

Our gadgets include:

- Interface to represent the edges.
- Five from B2CL.
- More because our geometry is more structured:
  - Turns in "wires".
  - Goal gadget
  - Makeup gadget with time-taking Red moves.
- Note: to win, Blue must move to all single-token stacks.

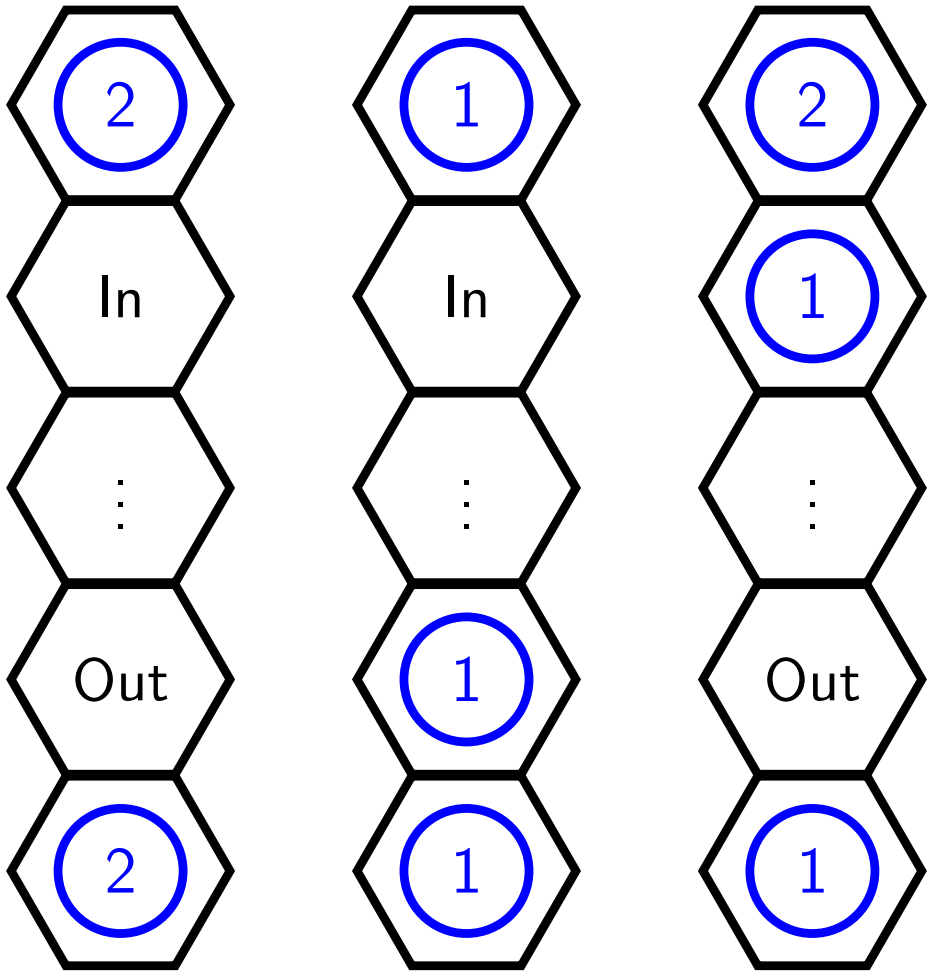
Reduction Gadgets

Variable:



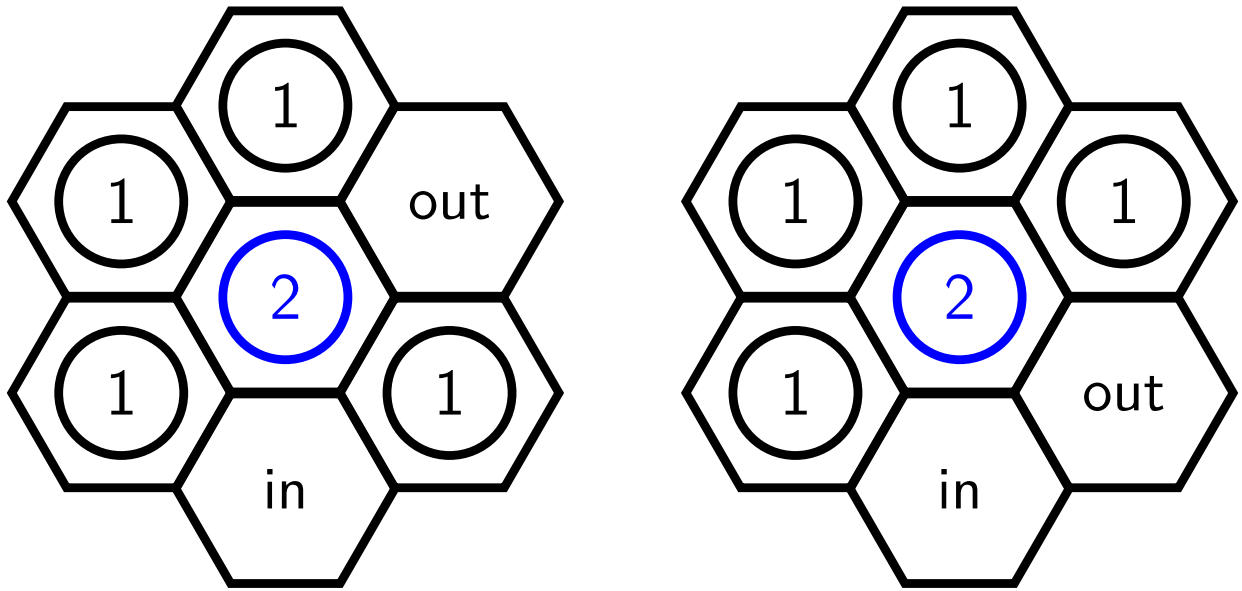
Reduction Gadgets

Interface between gadgets (Wire):



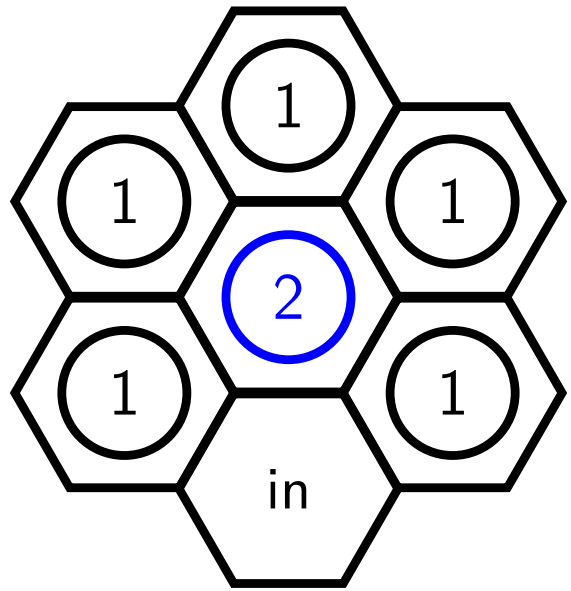
Reduction Gadgets

Wire Turns:



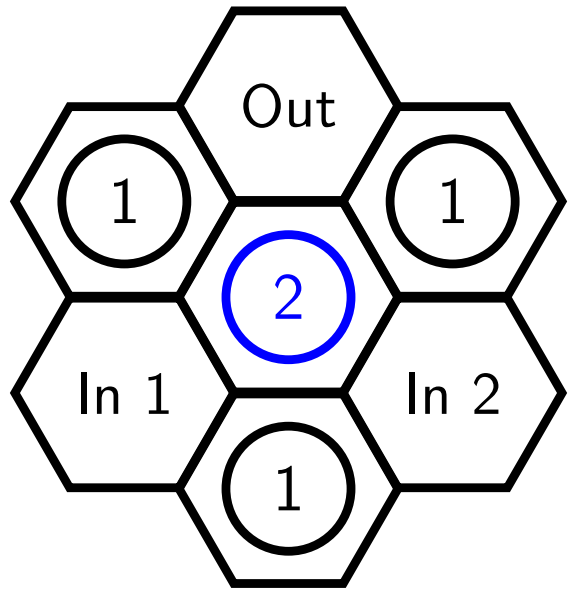
Reduction Gadgets

Goal:



Reduction Gadgets

Or:



Reduction Gadgets

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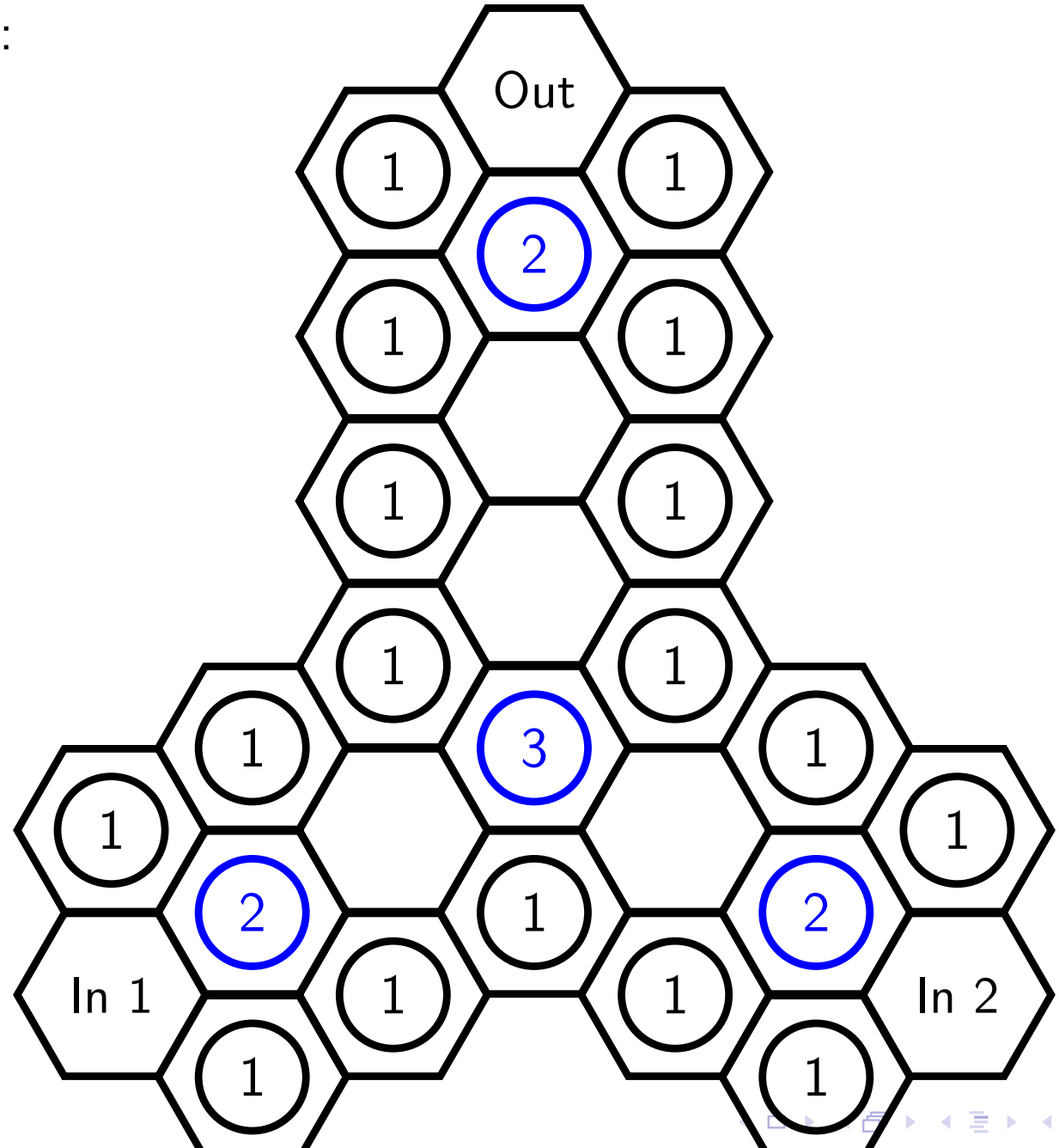
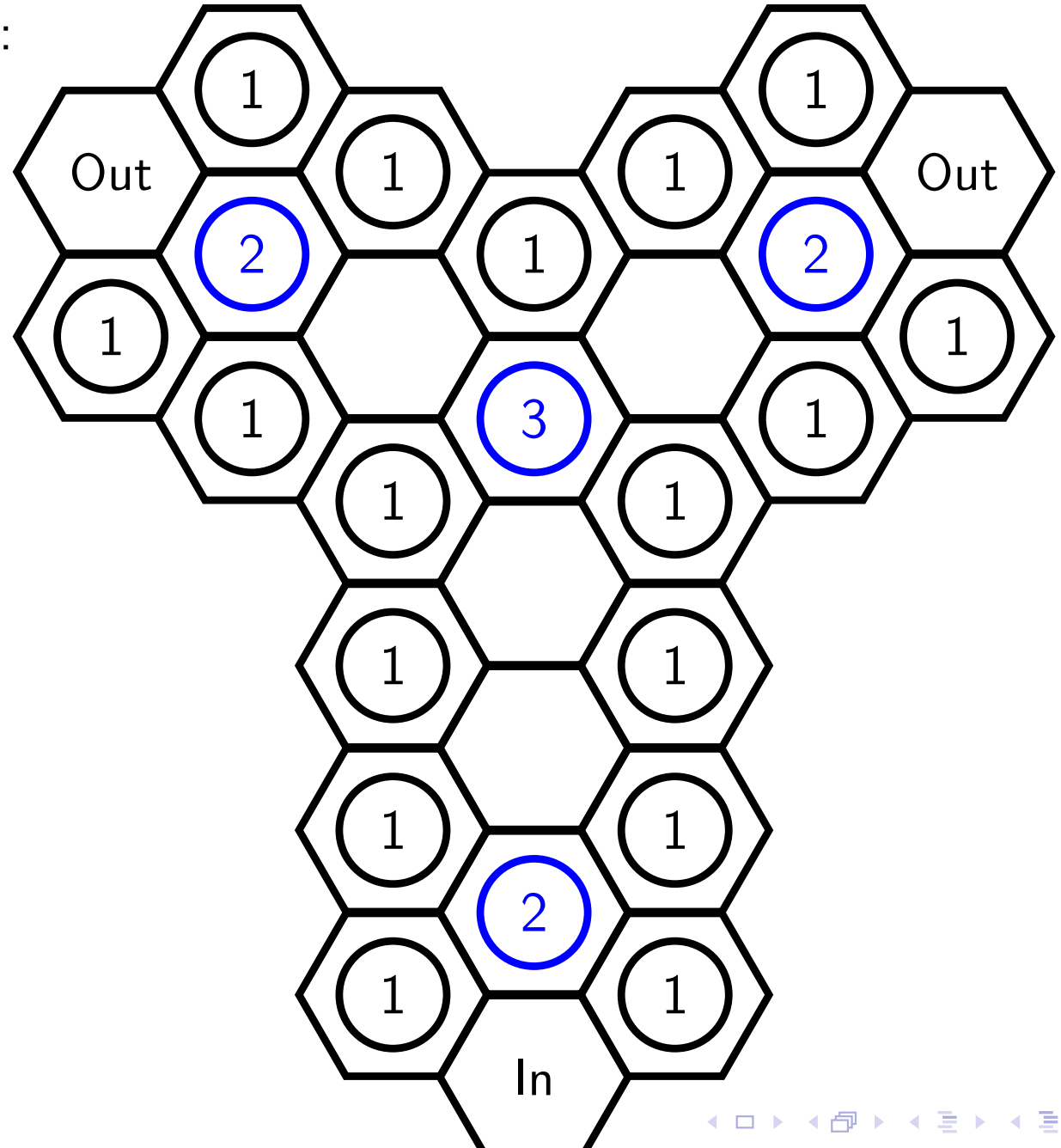


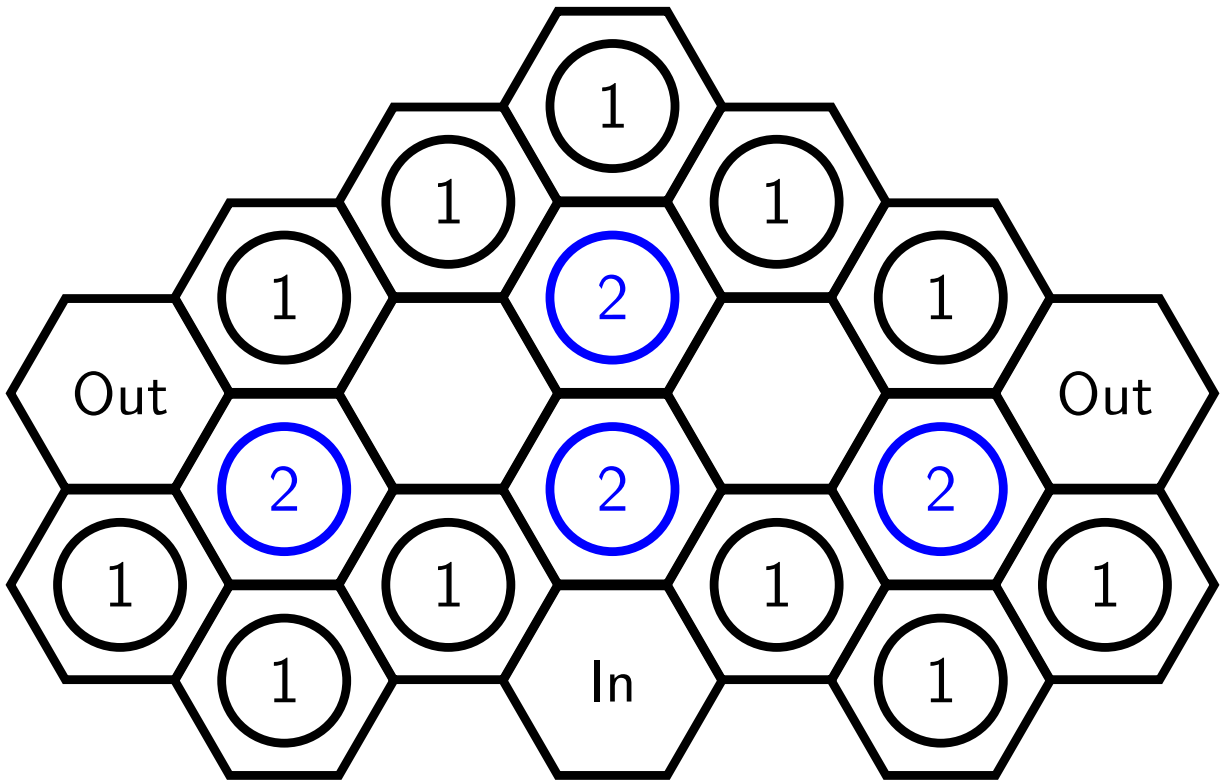
Reduction Gadgets

Fanout:



Reduction Gadgets

Choice:



Reduction Gadgets

- B2CL-needed gadgets:
  - Variable? ✓
  - Or? ✓
  - And? ✓
  - Fanout? ✓
  - Choice? ✓
- Other parts:
  - Wires? ✓
  - Turns? ✓
  - Goal? ✓
- Planarity? ✓B2CL gives a crossover gadget automatically!

The reduction works; Battle Sheep is PSPACE-complete.
(Likely) No polynomial-time solution for general positions.

Battle Sheep is PSPACE-complete

Conclusions!

- Battle Sheep is a fun game.
- Battle Sheep is PSPACE-complete: even more fun! (There are a lot of games like this.)
- You should attend Sprouts and play in the tournament! (Everyone)
- You should write a player for Battle Sheep for Sprouts! (Everyone)
- If you are an undergrad doing a project involving board games, you should give a talk at Sprouts!

Thank you!

Preprint: `http://arxiv.org/abs/2505.06414`

Sprouts: `https://kyleburke.info/sprouts/sprouts2026/`