Extroverted Play: Shang-Hua and Combinatorial Game Theory

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Story Time

- When first meeting, Shanghua and I brainstormed research ideas outside
- Shang-Hua continues to brag about LA's weather

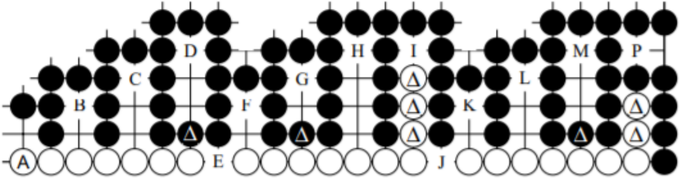
Story Time

- When first meeting, Shanghua and I brainstormed research ideas outside
- Shang-Hua continues to brag about LA's weather
- Shang-Hua being Shang-Hua, his first idea is try to work with the reconstruction conjecture
  - Do some ML with it, and if we are lucky, solve it!
  - I laugh nervously

CGT Background

- Eventually we end up working with **Combinatorial Games**
- Combinatorial games:
  - 2 players
  - Perfect information
  - No chance
  - Normal play convention (whoever can't move loses)

CGT Background

- Disjunctive Sum of games G and H
  - On a turn, pick either G or H and make a move on it
  - Game ends when no more moves can be played in either game
  - Written as $G + H$
- Main focus of Combinatorial Game Theory research
- Why?
- 

[1]

[1] **Taken from:** David Wolfe. "Go Endgames Are PSPACE-hard."

CGT Background

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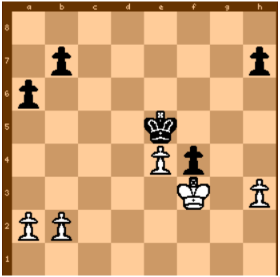
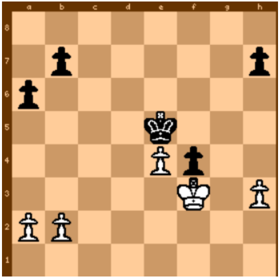
- Even when not obvious, games sometimes decompose
- 

- This is a real position from a 1929 game Schweda Vs Sika

CGT Background

- Gives an analysis tool
- 

- Early analysis of this game was based on a long brute-force explanation
- A mathmetician versed in CGT would just quickly prove the game on the right is $\downarrow + \downarrow + *$ and the game on the left is $\uparrow$ [Elkie 96]
- $\downarrow + \downarrow + * + \uparrow = \downarrow + *$
- This means that the first player to make a move wins

CGT Background

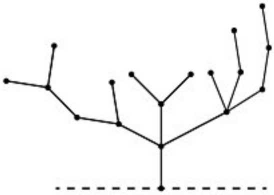
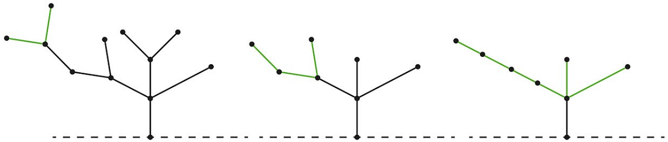
- Defining generalizations
-  [2]
- 

  Fig. 6 [3]

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[2] Taken from: `https://en.wikipedia.org/wiki/Hackenbush`

[3] Taken from: `https://www.wikiwand.com/en/Hackenbush`

CGT Background

- **Impartial** combinatorial games:
  - Same options for both players
  - Thus, the two possibilities are either who ever plays first wins, or whoever plays second wins
- Games that are not impartial are called **partizan**

Impartial CGT

- Nimbers: values for G, H used to determine who wins $G + H$
- $\{*0, *1, *2, \ldots\}$ except we simplify the first two: $\{0, *, *2, *3, \ldots\}$
- $G = 0 \Leftrightarrow$ who ever plays second on G wins
- $G = *k \Leftrightarrow$ whoever plays first on G wins
- $*k + *j = *(k \oplus j)$
  - $*7 + *7 = \ 0$
  - $*7 + *6 = \ *$

Geography

- (Directed) Geography
  - Position: Directed graph $G = (V, E)$ and a token on one vertex, v_1
  - Turn: move token to adjacent vertex, v_2. Then, delete v_1 and all incident edges
  - You lose if you can't move
- Even on planar bipartite graphs of degree 3, this game is PSPACE-complete. [Liechtenstein, Sipser 1980]
- Surprisingly, when played on an undirected graph, it is in P. [Frankel, Scheinermann, and Ullman 1993]

Geography

- So, something I realized for UNDIRECTED GEOGRAPHY:
  - There is an algorithm to find the solution
  - But, there was no algorithm for finding the nimber
- I couldn't think of another combinatorial game like this, so finding an algorithm would be interesting
- A small problem: I had to teach Shang-Hua what a nimber was!
  - This was a very nontrivial task
  - But of course, now Shang-Hua wants us to solve all computational problems involving nimbers!
- Me and Kyle a made secret pact: don't mention anything beyond impartial games
  - First his mind would be blown
  - Then he would ask us to solve every computational problem in CGT

An Algorithm?

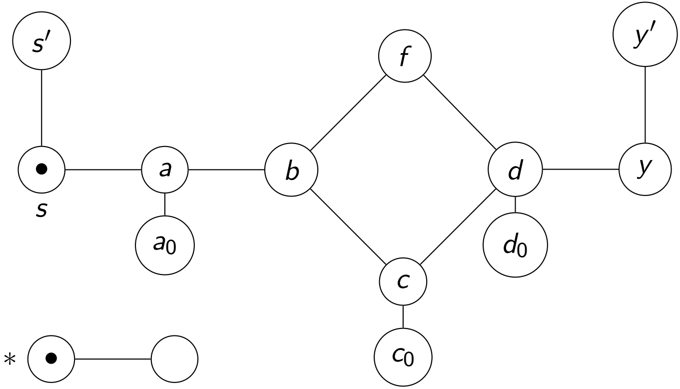
- Kyle comes up with an algorithm idea
- That night, at 9pm, I get a call from Shang-Hua
  - A "textbook proof"
- The next morning, I get another call
  - A "small hole" was found

Paper

- We eventually get a proof that finding the nimber is actually PSPACE-complete
- We just showed that there is a game that has a tractability gap between winnability and nimber identification!
- There are multiple papers gesturing at this question
- The proof is complicated
- We prepare a FOCS submission

Paper

- One week before submission, Kyle simplifies the proof to this



Paper

- Noooooooooooooooo
- Surely they won't accept such an easy proof
- We have to use it, since it improves the result to make it hard on planar bipartite graphs of degree 4
- Shang-Hua consoles me
- Then: when doing a more thorough literature review when writing, we briefly think we got scooped... in 1981

Paper

- It's hard to access paper
- Shang-Hua uses his "contacts" to get it

- Fortunately, it's just a paper that proves something weaker and alludes to our question

Formal Results

Our contribution was the following:

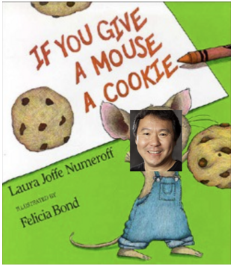
1. If each vertex in UNDIRECTED GEOGRAPHY game G has degree no more than 3, nimber(G) $\in P$
2. When relaxing the restriction to degree 4, distinguishing between $*$ and $*2$ in nimber(G) is PSPACE-complete, even on planar bipartite graphs.
3. For any pair of integers k and p, where $k, p > 0$, finding whether $G \in *k$ or in $G \in *p$ is PSPACE-complete.
4. We can use these results to finding the nimber for UNCOOPERATIVEUNO is PSPACE-complete

Partizan Games and Paint Can

- Partizan games: players may have different move options
- Describe as pairs in odd notation:
  - $G = \{ \ G^L \mid G^R \ \}$
  - E.g.: $\{ \ 0 \mid *, *2, *4 \ \}$
- From this you get number system with:
  - Integers. $1 = \{ \ 0 \mid \ \}, 5 = \{ \ 4 \mid \ \}, -220, \ldots$
  - Dyadic Rationals. $1/2 = \{ \ 0 \mid 1 \ \}, -47/64, \ldots$
  - Switches. $\pm 3 = \{ \ 3 \mid -3 \ \}, \pm 100, 2 \pm 5, \ldots$
- Remember:
  - Kept Shang-Hua in the dark about Partizan Games until we "finished" impartial results.
  - Not even integers!

Partizan Games and Paint Can

- Unfortunately, Shang-Hua is very thorough.
- Nearly scooped by Morris sums!
  - $\{\{5 \mid 3\} \mid \{-1 \mid -4\}\} + \{\{6 \mid 1\} \mid \{0 \mid -8\}\} + \cdots$
  - PSPACE-hard
  - Referenced in UNDIRECTED GEOGRAPHY paper.
- Shang-Hua got interested, and you know what happens when Shang-Hua gets interested...

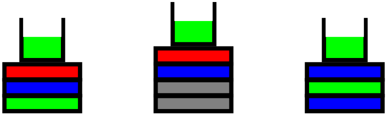


Partizan Games and Paint Can

- Started looking at temperature, the benefit gained by players moving first in each term.
- Recruited Svenja Huntemann, mathematician and temperature expert.
- Almost immediately, trail went lukewarm, tepid, even.
  - That's a good thing! Cold games: you don't want to play on them because it "costs" moves.
  - Tepid games: plays don't change overall temperature.
  - E.g.: Nimbers and games with only nimbers as options
  - $\{\ 0, *, *3 \mid 0, *, *4\ \} = *2$
  - $\{\ 0, *, *2 \mid 0, *2, *5\ \}$, not a nimber

Partizan Games and Paint Can

- What should we call these not-quite-nimbers?
- Silva, dos Santos, Neto, Nowakowski 2023: "Quasi-Nimbers"
- Paint Can:



- $\{\ 0, * \mid 0, *2\ \} + \{\ *2 \mid *3\ \} + \{\ 0, *, *2 \mid *\ \}$.
- Playable:
`https://kyleburke.info/DB/combGames/paintCan.html`

Partizan Games and Paint Can

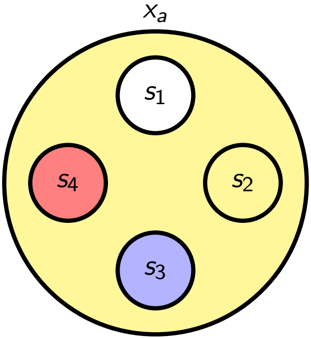
- "Quasi-Nimbers" in recent paper, but...
- 1982: Winning Ways used "superstars", but...
- 1976: On Numbers and Games used "superstars" for something else!
- Initially:
  - Quasi-Nimbers: Matt, Kyle, Svenja
  - Superstars: Shang-Hua
- However, Shang-Hua is persuasive.
- Soon:
  - Quasi-Nimbers:
  - Superstars: Shang-Hua, Matt, Kyle, Svenja ... and a bunch of CGT researchers!

Partizan Games and Paint Can

- What did we actually do?
  - PAINT CAN is NP-hard.
  - Fourth improvement on Morris sums
  - Sum of "shallow" games is still hard
  - Superstars: as close to nimbers as we can get.
- Reduced from Equal-Partitioned Multi-State XOR-Sat
- XOR Sat: $(x_1 \oplus x_2 \oplus x_3) \wedge (x_2 \oplus x_4 \oplus x_5) \wedge \cdots$
- (Solvable in P.)

Partizan Games and Paint Can

- Multi-State Variables



- x_a with four possible states: s_1, s_2, s_3, s_4. Set to s_2.
- Multi-State XOR-SAT:
  $(x_{1,s_2} \oplus x_{2,s_3} \oplus x_{1,s_1}) \wedge (x_{2,s_1} \oplus x_{3,s_1} \oplus x_{1,s_3}) \wedge \cdots$

Partizan Games and Paint Can

- Need to partition: half variables for each player.
- Equal-Partition Multi-State XOR-SAT (NP-hard)
  $(x_{1,s_2} \oplus x_{2,s_3} \oplus y_{1,s_1}) \wedge (x_{2,s_1} \oplus y_{3,s_1} \oplus y_{1,s_3}) \wedge \cdots$
- $\underbrace{(x_{1,s_2} \oplus y_{1,s_1})}_{1} \wedge \underbrace{(x_{2,s_1} \oplus y_{2,s_1})}_{2} \wedge \underbrace{(x_{1,s_1} \oplus y_{2,s_2})}_{4} \wedge \underbrace{(x_{1,s_1} \oplus y_{2,s_1})}_{8}$
- $G =$
  - $\{\ 0 \mid * \ \}$ (from y_1)
  - $+\{\ 0 \mid *10, *4\ \}$ (from y_2)
  - $+\{\ *, *12, *16 \mid 0\ \}$ (from x_1)
  - $+\{\ *2, *32 \mid 0\ \}$ (from x_2)
  - $+\ *15$ (from $2^m - 1$)
- True (X) goes 2^{nd}; activate all clauses (star term to zero)
- Always play on term with opponent zero.
- Big stars are winning responses to Y playing on star.

Shang-Hua's effect on CGT

- In 2018, Shang-Hua didn't know any Combinatorial Game Theory.
- Six years later:
  - Found first known solved impartial game with hard nimbers.
  - Found first known hardness in "quantumized" games
  - Found first reduction that preserves nimber values and defined computational classes around them.
  - Found hardness in sums of games with known values closest to nimbers.
- Shang-Hua is a superstar (and not a quasi-nimber)
  - We are very lucky to get to work with you, Shang-Hua.

Shang-Hua's effect on CGT

Thank you!

Binary Geography:
https://kyleburke.info/DB/combGames/twoBUG.html
Paint Can:
https://kyleburke.info/DB/combGames/paintCan.html