CGT Crash Course: Impartial Games

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Talk Plan

- What are impartial combinatorial games?
- Options and Game Trees
- Mex and Nimbers
- Game Sums and XOR

Impartial Basics

Impartial Combinatorial Games:

- Board/State/Game: "position".
- Two Players alternate turns.
- Impartial: both players can make the same moves ("options").
- No hidden information, no randomness.
- If you can't move, you lose. I.e. Last move wins. ("Normal" play)

We can describe positions (recursively) by their options.

Impartial Basics

Consider NIM:

- Position: a list (multi-set) of piles of objects (stones, **sticks**, coins, etc)
- E.g. / || \\\| has three piles of 1, 2, and 3 sticks.
- Each turn: pick a single pile and remove some (at least one) of the objects. (Yes, you can clear out an entire pile.)
- Can't pick an empty pile.

Options and Game Trees

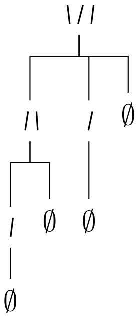
Think of an impartial position as a set of its options.

- $//\backslash = {}^*\{\ ||, \backslash, \emptyset\ \}$
- $|/ = {}^*\{\ /, \emptyset\ \}$
- $\backslash = {}^*\{\ \emptyset\ \}$
- So: $\backslash/\backslash = {}^*\{\ {}^*\{\ {}^*\{\ \emptyset\ \}, \emptyset\ \}, {}^*\{\ \emptyset\ \}, \emptyset\ \}$
- And, here, $\emptyset = {}^*\{\ \ \}$

This is *not* standard notation. (Craig and I used this specifically for impartial games in our text.)

Options and Game Trees

We also use non-standard notation for impartial game trees.
Here's the tree for 3 NIM sticks:



Outcome Classes

We can describe *winnability* of positions using *outcome classes*.

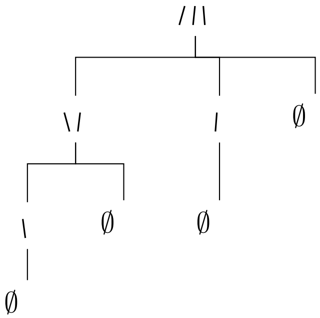
- $\mathcal{N}$, "Fuzzy": All positions where $\mathcal{N}$ext player has winning strategy.
- $\mathcal{P}$, "Zero": All positions where $\mathcal{P}$revious player has winning strategy.

We can recursively describe how to find the outcome classes.

- If G has an option $\in \mathcal{P}$, then $G \in \mathcal{N}$.
- If all of G's options are $\in \mathcal{N}$, then $G \in \mathcal{P}$. Note: terminal positions are trivially in $\mathcal{P}$.

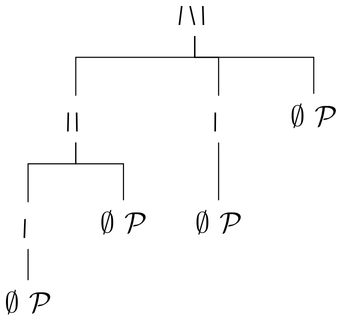
Outcome Classes

Let's find the outcome class of |/\:



Outcome Classes

Let's find the outcome class of / \ |:



Outcome Classes

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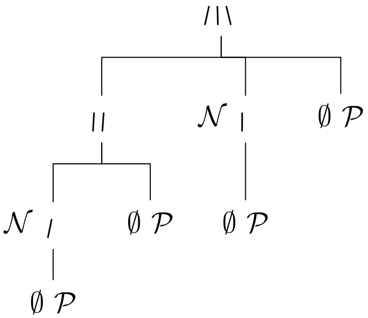
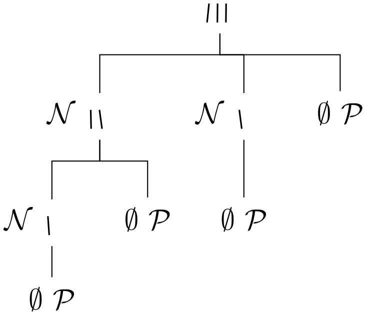


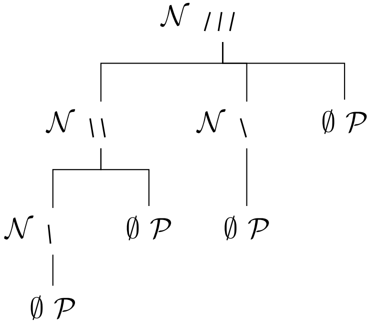
Outcome Classes

Let's find the outcome class of $\backslash / |$:



Outcome Classes

Let's find the outcome class of |||:



In any single NIM heap, it will always be in $\mathcal{N}$ so long as there is at least one item, $\mathcal{P}$ otherwise.

Game Sums

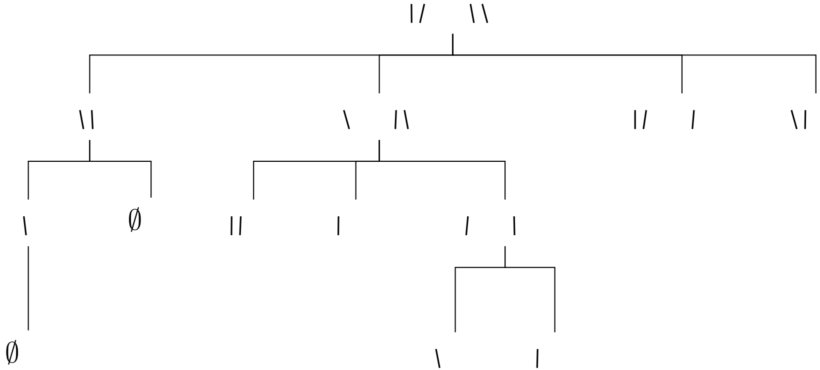
Since single-heap NIM games are a bit boring, let's add two together!

|| \\ is a position with two heaps of two items each.

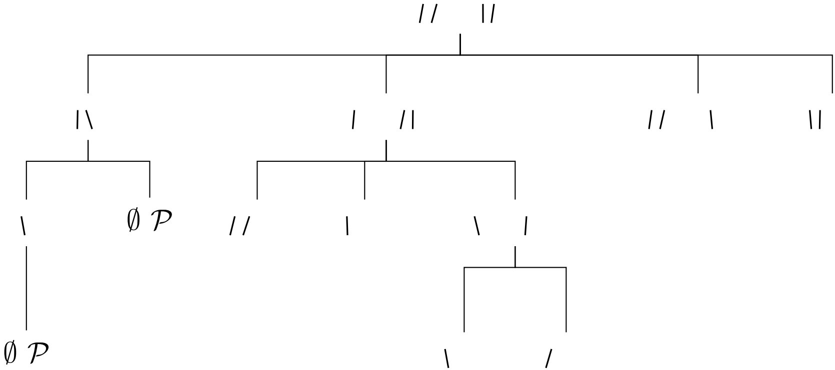
- Each turn, the current player makes a move on one of the heaps.
- Sum of games: \\ || = /\ + /|
- In general, on $G = G_1 + G_2$, the current player makes a move on one of G_1, G_2.

Let's take a closer look at /| ||. Next diagram has many shortcuts.

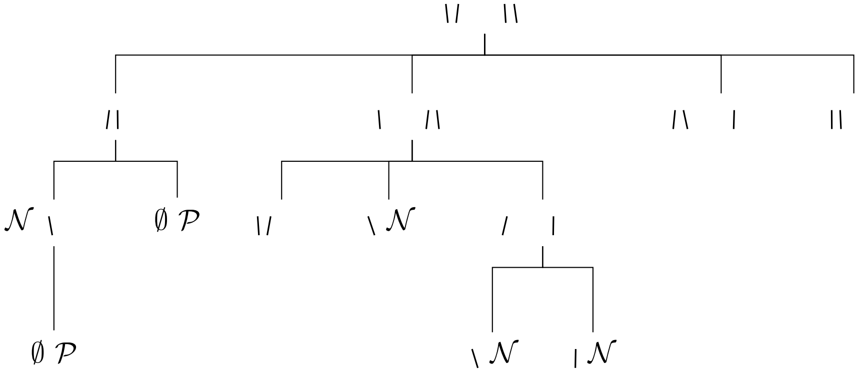
Game Sums



Game Sums

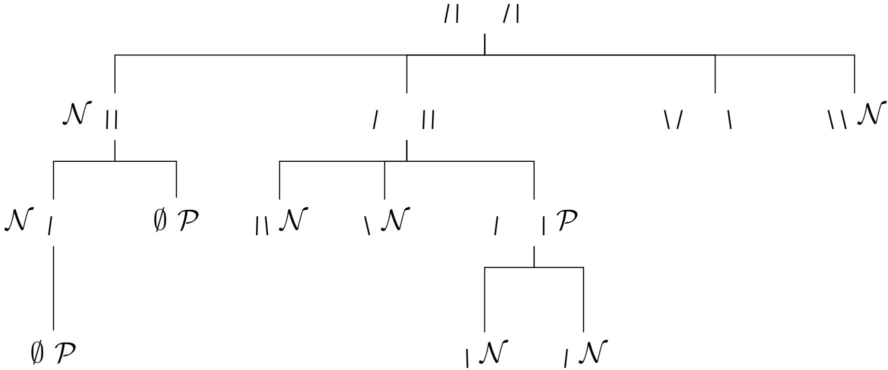


Game Sums

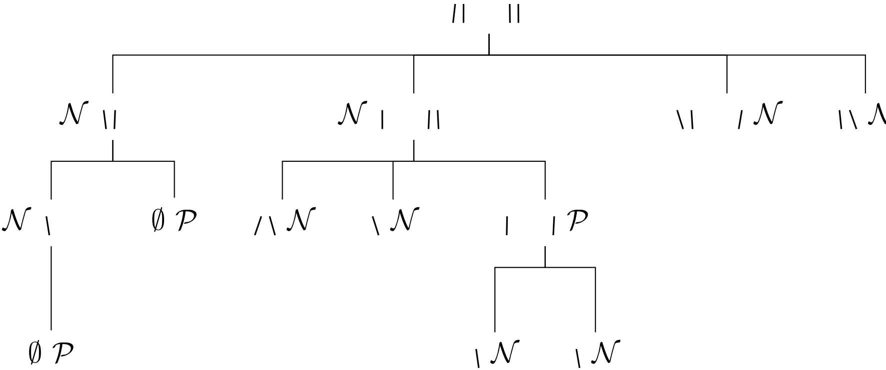


\|/ |\\
/|\\
\\ /|\\
|\\ |
||
$\mathcal{N}$ \\
$\emptyset$ $\mathcal{P}$
\|/
\\ $\mathcal{N}$
/ |
$\emptyset$ $\mathcal{P}$
\\ $\mathcal{N}$
| $\mathcal{N}$

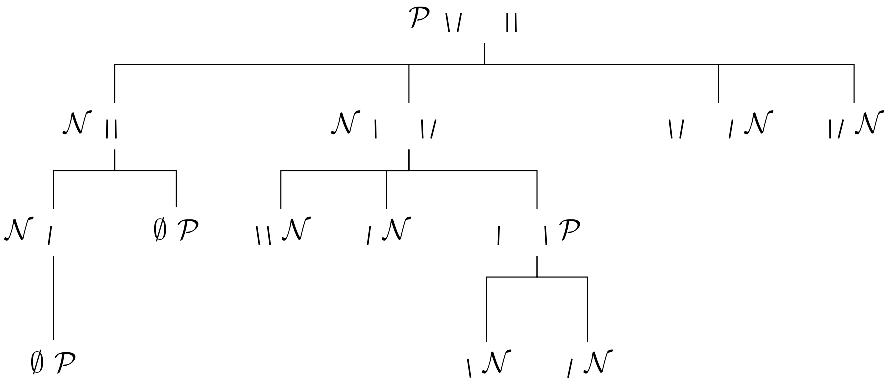
Game Sums



Game Sums



Game Sums



So, /| |/ $\in \mathcal{P}$

Game Sums

Any two NIM piles of the same size will sum to $\mathcal{P}$, but it's not always so simple...

- In general, $\mathcal{P} + \mathcal{P} = \mathcal{P}$.
- $\mathcal{N} + \mathcal{P} = \mathcal{N}$.
- $\mathcal{P} + \mathcal{N} = \mathcal{N}$.
- However, $\mathcal{N} + \mathcal{N}$ can go either way.
  - |\ \\| $\in \mathcal{P}$ (piles are equal), but
  - | \\/| $\in \mathcal{N}$ (piles are different).

What about three or more piles? Those game trees are huge...

Nimbers

Good news: there are values! (Nimbers, AKA Grundy values)
They are: $*0, *1, *2, *3, *4, *5, \ldots$. (Not quite $\mathbb{N}$.)
First two simplified: $0, *, *2, *3, *4, *5, \ldots$.

- $\emptyset = *0 = 0$
- \ $= *1 = *$
- / \ $= *2$
- \ / / $= *3$
- \ / / \ $= *4$
- $\vdots$
- $G = 0 \Leftrightarrow G \in \mathcal{P}$

Multi-heap NIM games are also equivalent to a nimber! We can do some math to find them.

Nimbers

Q: How do we find the nimber of a sum of NIM games?
A: XOR the nimbers together! $*k + *m = *(k \oplus m)$

$$\begin{aligned} *6 + *5 &= *(110 \oplus 101) \\ & \quad\ \ 110 \\ & \underline{\oplus\ 101} \\ &= \quad 011 \\ &= \quad *3 \end{aligned}$$

- $*8 + *7 = *15$
- $* + * = 0$. In fact, $*k + *k = 0$
- $*10 + *13 = *7$

Nimbers

What is the outcome class of / |\|/ |\||| ?

$$
\begin{aligned}
\text{\textbackslash} \quad |\backslash// \quad |/|\backslash\backslash &= * + *4 + *5 \\
&= *(001 \oplus 100 \oplus 101) \\
& \quad\ \ 001 \\
& \quad\ \ 100 \\
& \underline{\oplus\ 101} \\
&= \quad 000 \\
&= \ 0
\end{aligned}
$$

So, \   \\\| /||\| $\in \mathcal{P}$.

Nimbers

Sprague-Grundy Theory : Every impartial game has a nimber value

We can find the nimbers with mex (minimum excluded value):
$mex\,(S \subset (\mathbb{N} \cup \{0\})) =$ smallest $x \in \mathbb{N} \cup \{0\}$ where $x \notin S$.

- $mex\,(\{0, 1, 2\}) = 3$
- $mex\,(\{0, 2\}) = 1$
- $mex\,(\{1, 2, 5\}) = 0$

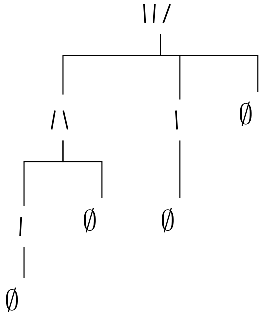
Nimbers

To find k so $G = *k$...
Find nimbers of options $G = {}^*\{ *a, *b, *c, *d, *e, \dots \}$
Then $k = mex\left(\{a, b, c, d, e, \dots\}\right)$.

- ${}^*\{ 0, *, *2, *3 \} = *4$
- ${}^*\{ \ \} = 0$
- ${}^*\{ *, *2, *3, *5 \} = 0$
- ${}^*\{ 0, *2, *4, *6 \} = *$

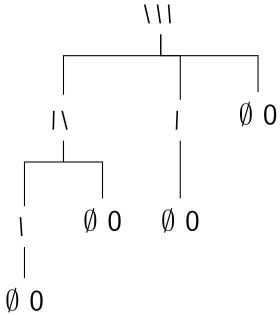
Nimbers

Let's double-check the nimber of \ | /:



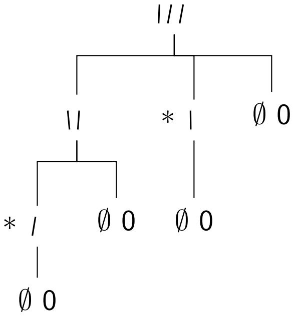
Nimbers

Let's double-check the nimber of ∣∣∣:



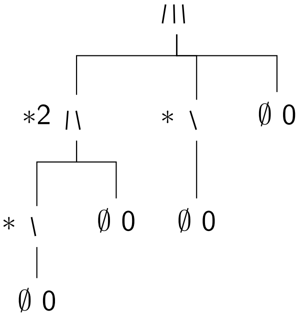
Nimbers

Let's double-check the nimber of \\\:



Nimbers

Let's double-check the nimber of / / \:



Nimbers

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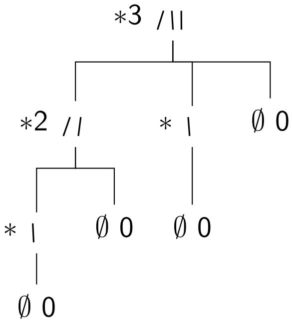
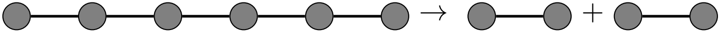


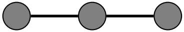
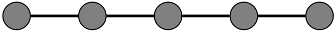
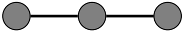
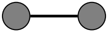
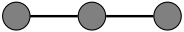
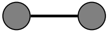
Any NIM heap with k items equals $*k$.

Arc Kayles

Let's try this all out on another ruleset: ARC KAYLES.

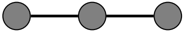
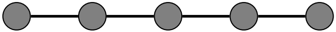
- An ARC KAYLES position consists of an undirected graph, and a move consists of picking an edge and removing it as well as both vertices and their incident edges.
- For example, taking the middle edge from .
- Let's use Sprague-Grundy theory to find the outcome class of 

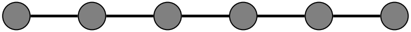
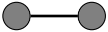
Arc Kayles

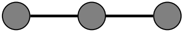
-  $= {}^{*}\{\ \ \} = 0$
-  $= {}^{*}\{\ \emptyset\ \} = {}^{*}\{\ 0\ \} = *$
-  $= {}^{*}\{$ ,  $\} = {}^{*}\{$  $\} = {}^{*}\{\ 0\ \} = *$
- 
  $= {}^{*}\{$ ,  $+$  $\} = {}^{*}\{\ *, 0 + 0\ \} = {}^{*}\{\ *, 0\ \} = *2$
- 
  $= {}^{*}\{$ ,  $+$  $\} = {}^{*}\{\ *, 0 + *\ \} = 0$
- 
  $= {}^{*}\{$ ,  $+$ ,  $+$
   $\} = {}^{*}\{\ *2, 0 + *, * + *\ \} = {}^{*}\{\ *2, *, 0\ \} = *3$

Arc Kayles

Summarized:

-  $= 0$
-  $= *$
-  $= *$
-  $= *2$
-  $= 0$
-  $= *3$

 $+$  $= *3 + * = *2 \in \mathcal{N}$

The winning move is to:   $+$ 

Impartial CGT Summary

- Impartial Game positions have nimber values, $0, *, *2, *3, \ldots$
- $0 \in \mathcal{P}$, $\forall k \in \mathbb{N} : *(k+1) \in \mathcal{N}$
- The nimber value of G is the mex of it's option nimbers.
- The value of $G + H$ is the binary-xor of their nimbers.

Additionally:

- Even though this gives us a lot of power to evaluate impartial games, it can still be quite hard to find the (nimber) value if the game tree is large and complex.
- Some rulesets (or rulesets on specific position families) might have values only 0 and $*$, making them They-Love-Me-They-Love-Me-Not positions.

CGT Crash Course: Partizan Games

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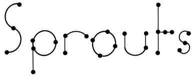


Talk Plan

- What are partisan combinatorial games?
- Notation, Outcome Classes, and Game Trees
- Inequalities
- Integers and Numbers
- Switches and... more (We'll skip a lot)

Partisan Basics

- Just like impartial games, but players can make different moves.
- Players: Left vs Right
- Naming Conventions:
  - bLue, bLack, Feminine prounouns $\rightarrow$ Left.
  - Red, white, Masculine pronouns $\rightarrow$ Right.
- Notation: $G = \left\{ \text{Left's options, } G^L \;\middle|\; \text{Right's options, } G^R \right\}$.
- So $* = \{\ 0 \mid 0\ \}$ and $*2 = \{\ 0, * \mid 0, *\ \}$.

Partisan Basics

Old outcome classes:

- $* = \{ 0 \mid 0 \} \in \mathcal{N}$
  $\{ 0, *, *2 \mid 0 \} \in \mathcal{N}$
- $\{ * \mid *2 \} \in \mathcal{P}$
- But $\{ 0 \mid * \} \notin \mathcal{N} \cup \mathcal{P}$ Left wins no matter who goes first.

New outcome classes:

- $\{ 0 \mid * \} \in \mathcal{L}$ "Positive"
- Similarly, $\{ *2 \mid 0 \} \in \mathcal{R}$, "Negative", Right wins no matter who goes first.

Partisan Basics

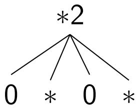
In other words:

- $\mathcal{N}$ is the set of positions where whoever goes first has a winning strategy.
- $\mathcal{P}$ is the set of positions where whoever goes second has a winning strategy.
- $\mathcal{L}$ is the set of positions where Left always has a winning strategy.
- $\mathcal{R}$ is the set of positions where Right always has a winning strategy.

The first two are independent of players' identities. The second two are independent of who is going first.

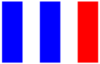
Partisan Basics

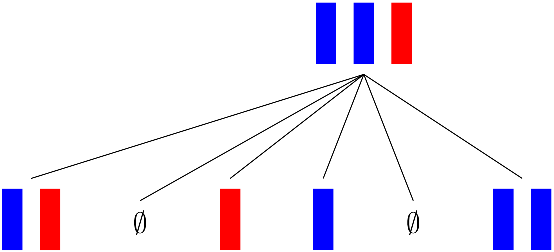
Game trees also look a bit different! Left options go left, Right options go right. (All straight lines.)



Partisan Basics

Consider the ruleset Toppling Dominoes.

- Position: row of Blue and Red dominoes, e.g. 
- Options: a player chooses one of their dominoes, and knocks it in either direction.

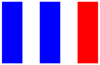
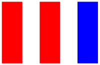


You might already see that some options are better than others...

Partisan Basics

With Partizan games, we need negatives.

- Informally: $-G$ means we just swap the roles of players in G

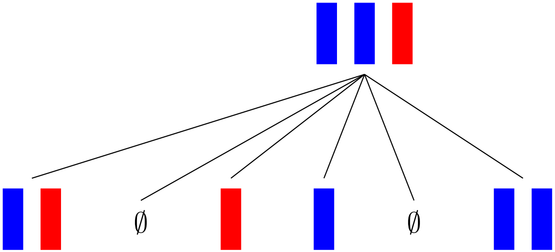
  $-$  $=$ 

- Formally, we flip sides and recursively negate the options.

  $-\{\ a, b \mid x, y, z\ \} = \{\ -x, -y, -z \mid -a, -b\ \}.$

Partisan Basics

Let's look at our prior game tree:



It would be helpful to get rid of some of these options.

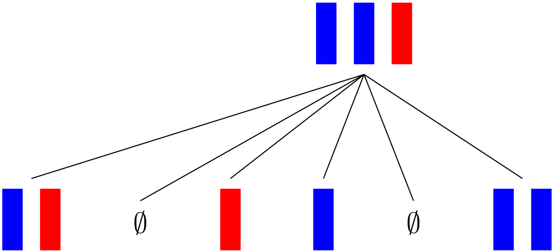
Inequalities

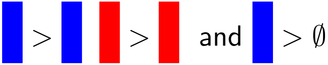
$G > H$ means G is *always* better than H for Left, and H is *always* better than G for Right

- In other words, $G - H > H - H = 0$.
- Or, $G - H \in \mathcal{L}$
- $G > 0 \Leftrightarrow G \in \mathcal{L}$
- $G < 0 \Leftrightarrow G \in \mathcal{R}$
- $G = 0 \Leftrightarrow G \in \mathcal{P}$
- But... we need a fourth category: $G \parallel 0 \Leftrightarrow G \in \mathcal{N}$
  (E.g. $*$, which is neither always better for Left or better for Right than 0.)

Inequalities

If players have multiple options, they might be dominated...

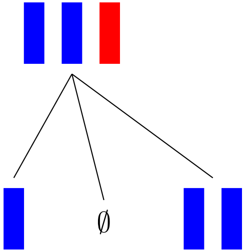




So Left will always choose the single blue domino over any of the others. We can remove them from consideration.

Inequalities

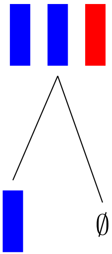
If players have multiple options, they might be dominated...

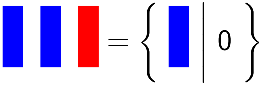


Also,  $> \emptyset$.
We can simplify again! For Right we remove the greater options...

Inequalities

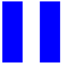
If players have multiple options, they might be dominated...



In other words, 

Numbers

What is the value of  ?

- In our notation,  $= \{\ 0 \mid\ \}$
- Left has one move to reach zero, Right has no options.
- "Left has a free move"
- $\{\ 0 \mid\ \} = 1$, and  $= \{\ \mid 0\ \} = -1$
-  $= \{\ 1 \mid\ \} = 2$
-  $= \{\ 4 \mid\ \} = 5$
- $\{\ k \mid\ \} = k + 1, \ (k \in \mathbb{N} \cup \{0\})$
- $\{\ \mid -k\ \} = -(k + 1)$
- Integers ✓

Numbers

What if the integers are on the wrong side?

- $\{ -2 \mid \ \} = ?$
- $-2 < 0$, so that is a losing option for Left.
- Neither Left nor Right has a winning move. Thus, the game is in $\mathcal{P}$.
- $\{ -2 \mid \ \} \in \mathcal{P} = \{0\}$, so $\{ -2 \mid \ \} = 0$.
- Same is true for $\{ \ \mid 8 \} = 0$

Numbers

We got $\{ x \mid \ \}$ and $\{ \ \mid x \}$. What about $\{ x \mid y \}$?

- If $x < y$, then $\{ x \mid y \} =$ the Simplest Number, k, between x and y. So $x < k < y$.
  - If there are any integers between x and y, then k is the one with lowest absolute value.
    - E.g. $\{ -2 \mid 10 \} = 0$
    - $\{ -10 \mid -7 \} = -8$
    - $\{ 0 \mid 4 \} = 1$
  - If there *aren't* integers between x and y, then k is the dyadic rational with smallest denominator.
    - "Dyadic Rational" sounds super complicated, but it isn't!
    - Denominator is a power of 2. (E.g. 2, 4, 8, ...)
    - Numerator is an odd integer.
    - E.g.: $\frac{3}{2}, \frac{15}{16}, -\frac{5}{4}$

Numbers

Dyadic Rationals Examples

- $\{ 5 \mid 6 \} = \frac{11}{2}$
- $\left\{ \frac{1}{2} \;\middle|\; \frac{3}{4} \right\} = \frac{5}{8}$
- $\left\{ -4 \;\middle|\; -\frac{7}{2} \right\} = -\frac{15}{4}$
- (Dyadic) Rationals ✓

(We need infinite sums, long games, or loopy games to get the rest of the real numbers.)

Numbers

Addition on Numbers works exactly like we want.

- $\{\ 4 \mid\ \} + \{\ \mid -6\ \} = 5 - 7 = -2$
- $\left\{\ 0 \;\middle|\; \frac{3}{4}\ \right\} + \{\ \mid -5\ \} = \frac{1}{2} - 6 = -\frac{11}{2}$
- $\{\ 0 \mid 1\ \} + \left\{\ \frac{7}{8} \;\middle|\; 1\ \right\} = \frac{1}{2} + \frac{15}{16} = \frac{23}{16}$

Switches

What about $\{ x \mid y \}$ where $x \geq y$?

- Hot games: games where you want to make a move.
- On $\{ 2 \mid -2 \}$, it's really good to go first! We call this ± 2
- $\{ 4 \mid 0 \} = \pm 2 + 2 = 2 \pm 2$.
- "Switches": $a \pm b$
- $\{ x \mid y \} = (x + y)/2 \pm (x - y)/2$

Switches

TOPPLING DOMINOES has Switches!

-  $= \{\, 0 \mid -4 \,\} = -2 \pm 2$
-  $= \{\, 2 \mid -6 \,\} = -1 \pm 4$

Switches

What about $\{ x \mid x \}$?

- $\{ 0 \mid 0 \} = * = \pm 0$
- $\{ x \mid x \} = x \pm 0 = x + *$

Switches

Adding Switches

- $a \pm b + c \pm d = (a + c) \pm b \pm d$
- Can we simplify the $\pm b \pm d$ part?
- $\pm b \pm d = 0 \Leftrightarrow b = d$ Otherwise we can't simplify!
- "Hotter" switches are more important, so common to list highest to lowest.
- $5 \pm 100 \pm 4 + -6 \pm 4 \pm 2 = -1 \pm 100 \pm 2$
- $5 \pm 100 \pm 4 + -6 \pm 5 \pm 2 = -1 \pm 100 \pm 5 \pm 4 \pm 2$

Beyond

There's a lot more!

- Infinitiesimals. E.g. $\{\ 0 \mid * \ \}$
- Switch-like games. E.g. $\{\ \{\ 4 \mid 2\ \} \mid \{\ -1 \mid -100\ \}\ \}$
- Reversible options
- Lots of things!

Resources:

- Winning Ways (classic text)
- Lessons in Play
- Combinatorial Game Theory (Siegel) also CGSuite: `http://cgsuite.sourceforge.net/`
- Playing with Discrete Math (our text, `http://kyleburke.info/CGTBook.php`)

Thank you!

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