Additional Computational Complexity of games Considerations

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Talk Plan

- Algorithmic CGT (ACGT) in general
- Complexity limitations
- Constraint Logic Power
- Value/Addition Complexity

Note: interruptible!

ACGT

- Algorithmic CGT: [DH 2009[1]]
  - How hard is it to play a ruleset R perfectly?
  - Main question: Does Left have a win going first on G?
  - Equivalent: $G \in^? \mathcal{L} \cup \mathcal{N}$
  - We treat rulesets as computational problems with assumed outcome-class-based question.
- Recursive question: $G' \in^? G^{\mathcal{L}} : G' \in \mathcal{L} \cup \mathcal{P}$
- Solve "how to win" question with "can I win" questions.
- Need: polynomial fanout. (Common)
- Polynomial tree height + polynomial fanout $\Rightarrow R \in PSPACE$

[1] "Playing Games with Algorithms: Algorithmic Combinatorial Game Theory", E. Demaine, R. Hearn, 2009

ACGT

- Assumed outcome-class-based ("winnability") questions:
- "Next player can win?" with role chosen/assumed. (E.g. True in Boolean Formula)
  - $G \in^? \mathcal{L} \cup \mathcal{N}$ ✓, $G \in^? \mathcal{R} \cup \mathcal{N}$ ✓,
  - $G \in^? \mathcal{L} \cup \mathcal{P}$ ✓, $G \in^? \mathcal{R} \cup \mathcal{P}$ ✓
  - $G \in^? \mathcal{L} \cup \mathcal{R}$ $\oslash$, $G \in^? \mathcal{N} \cup \mathcal{P}$ $\oslash$
- Similar: $G \in^? \mathcal{N}$, $G \in^? \mathcal{L}$, $G \in^? \mathcal{R}$, $G \in^? \mathcal{P}$ (But, seldom explicitly asked)
  - Impartial games get this automatically.
    $G \in^? \mathcal{L} \cup \mathcal{N}$ is hard $\Rightarrow$ $G \in^? \mathcal{N}$ is hard

ACGT

- If ruleset R ($G \in R$) is symmetric ($\forall G : -G \in R$ and $f(G) = -G$ is efficient) then two questions are complementary
  - If I can efficiently determine $\forall G$ whether $G \in \mathcal{L} \cup \mathcal{N}$,
  - Then I can determine whether $-H \in \mathcal{L} \cup \mathcal{N}$,
  - Which is the same as $H \in \mathcal{R} \cup \mathcal{N}$,
  - Which, if false, means $H \in \mathcal{L} \cup \mathcal{P}$.
- Most rulesets are symmetric. (Not always e.g. HNEFATAFL)

ACGT

EXPTIME

- Can be solved in exponential time.
- $\mathrm{PSPACE} \subseteq \mathrm{EXPTIME}$
- Actually known: $\mathrm{P} \neq \mathrm{EXPTIME}$
- Stronger results! Harder reductions. Fewer rulesets?
- Usually loopy games. Tree height no longer polynomial.
- hard vs. complete:
  R is X-complete means R is X-hard and $\mathrm{R} \in \mathrm{X}$.
- $\mathrm{EXPTIME}$-hard actually means need exponential time.
- Some PSPACE-hard results could be improved to PSPACE-complete or $\mathrm{EXPTIME}$-complete. (More options.)

ACGT

- Fun research problems!
- Accessible for advanced undergrads.
- ACGT also includes finding algorithms to solve games in polynomial time!

ACGT Limitations

Image space of the reduction.

- $Img(f) = \{H \mid H = f(G), G \in \text{Boolean Formula}\}$
  "Image" or "Image positions"
- Reachable? What are R's starting positions?
- Actual played/implemented geometry?
- Likely to encounter? Distance to played positions? Measurable?
- Enormous size blowup.

ACGT Limitations

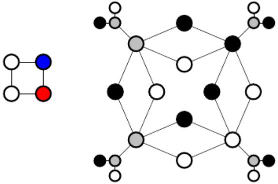
Image not always reachable.

- Is there a sequence of plays from starting positions to image?
- Maybe not. E.g. Col w/ two adjacent same-color vertices
- Papers published with this all the time. (Probably me!)
- ... and that's okay! Still means no algorithm $A(G)$ that returns whether $G \in \mathcal{L} \cup \mathcal{N}$ for general positions G.
- Rulesets are (probably) 4-tuples: $(States, optns_L, optns_R, Start)$
- Enforced that *States* must be followers of *Start* elements?
- Not addressed: usually obvious, or not mentioned, or *Start* not defined.

ACGT Limitations

Do image positions use actual position geometry?

- Does the image exist in implemented games?
- E.g. NoGo on grids[2], but reduced graphs aren't grids.



- I publish these often. Never seen NoGo on general graphs.
- Cool: general graphs → planar graphs → hex grid → grid
- Maybe you can show NoGo is hard on grids!

[2] That's what I did: http://kyleburke.info/DB/combGames/noGo.html

ACGT Limitations

Image positions likely in (good) play?

- Will the image exist in played game? (No!)
- Dream world: "Magnus Karlsen's move at this event in 2022 created result of reduction from some G."
- Other problems:
  - Would be easy for one player to avoid (and avoidance would be beneficial)
  - High concentration of placed pieces "away" from action.
- Still okay to publish. Criticism is rare. Can we even measure "distance" to played positions.
- Reduction is still evidence that game is likely hard.

ACGT Limitations

Image boards are enormous.

- Similar to arguments about Galactic algorithms.
- No one is going to play ATROPOS on 30×30 boards.
- This is all hardness reductions! Always a blowup.
- Yes, submit your result for publication! Please!

There is more to criticize than normal hardness results. I want to know what you found out!

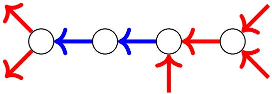
Constraint Logic

- Sometimes you can't get all the gadgets
- E.g. reducing from BOOLEAN FORMULA, get:
  - Variable and Wire gadgets
  - And and Or gadgets
  - Split and Join wire gadgets
  - But no negations.
- Try CONSTRAINT LOGIC [HD 2009[3]]
- Generalized directed graph game with flippable edges

[3] "Games, Puzzles, and Computation", R. Hearn, E. Demaine, 2009.

Constraint Logic

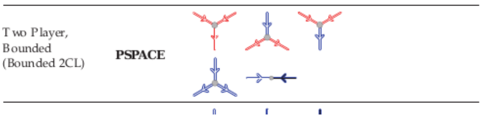
- Constraint Logic:
  - Played on a directed graph
  - Arcs have weights 1 and 2.
  - Each vertex must always have "inflow" of at least 2.



  - Goal edges: flip to win.
  - Multiple versions for complexity classes.
  - Two-player version: Edges belong to one player each.
  - Non-loopy: each edge can only be flipped once.
  - "Bounded 2-Player Constraint Logic", B2CL

Constraint Logic

- B2CL is PSPACE-complete
- CONSTRAINT LOGIC: circuit-like game, without negations
- Simplifies PSPACE reductions:
  - Only need up to degree 3 vertices.
  - Don't need to implement all vertex combinations. For B2CL:



    Like gates: inputs on bottom, outputs on top
    Names: CHOICE, AND, SPLIT, OR, VARIABLE
  - Skip B2CL details completely, and just implement those 5 gadgets and Goal edges (and maybe wires)
- Planarity is free!

Constraint Logic

- B2CL used on many combinatorial games
  - Amazons, Konane, Cross-Purposes[HD 2009[4]]
  - Planar Col, Fjords[BH 2019[5]]
  - Forced-Capture Hnefatafl[BT[6]]
- Start with book: *Games, Puzzles, and Computation*, Demaine and Hearn.

[4] "Games, Puzzles, and Computation", R. Hearn, E. Demaine, 2009.

[5] "PSPACE-complete two-color planar placement games", KB, R. Hearn, 2019

[6] "Forced-Capture Hnefatafl", KB, C. Tennenhouse, preprint

Addition Complexity

- Main CGT: Can we evaluate position by evaluating parts?
  - Disjunctive sum: Break into independent parts.
  - Evaluate those parts
  - Add the values together to get total value.
- E.g. $2 + * + \pm 1 \in \mathcal{L}$
  $-6 + *4 \pm 10 \in \mathcal{N}$
- Pretty easy to add these values together.

Addition Complexity

- When is it hard to add values together?
- One "level" from switches: Switch-like games
- E.g.: $\{ 4 \pm 1 \mid -5 \pm 4 \} = \{ \{ 5 \mid 3 \} \mid \{ -1 \mid -9 \} \}$
  $\{ \{ 4.5 \mid -2.5 \} \mid \{ -1 \mid -5 \} \} = \{ 1 \pm 3.5 \mid -3 \pm 2 \}$
- Sums are complicated.
  $\{ 4 \pm 1 \mid -5 \pm 4 \} + \{ 1 \pm 3.5 \mid -3 \pm 2 \}$
- "Shallow sums" shown to be intractible!
- Morris 81[7] → Yedwab 85[8] → Moews 93[9] → Wolfe 2000[10]
- Sums of $\{ x \mid \{ y \mid z \} \}$ are PSPACE-complete

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[7] "Playing disjunctive sums is polynomial space complete.", FL Morris, 1981
[8] "On playing well in a sum of games", LJ Yedwab, 1985
[9] "On some combinatorial games connected with Go", D Moews 1993
[10] "Go endgames are pspace-hard", D Wolfe, 2000

Addition Complexity

- Sums of $\{\ x \mid \{\ y \mid z\ \}\ \} \rightarrow \mathrm{PSPACE}$-complete
- Polynomial number of summands
- Hard games can be either: single deep games or sums of shallow games.

Addition Complexity

- From before: complexity of determining outcome class.
- UNDIRECTED GEOGRAPHY $\in P$ (Neat matching algorithm) [FSU 1993[11]]
- Even deeper: complexity of determining value.
- Deciding whether $G \in$ UNDIRECTED GEOGRAPHY $= *$ PSPACE-complete [BFT 2022[12]]

[11] "Undirected edge geography", A. Fraenkel, ER Scheinerman, D. Ullman, 1993

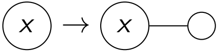
[12] "Winning the War by (Strategically) Losing Battles: Settling the Complexity of Grundy-Values in Undirected Geography", KB, M. Ferland, S. Teng, 2022

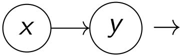
Addition Complexity

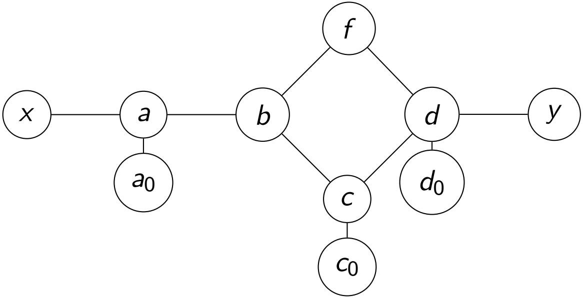
- How? Reduced GEOGRAPHY $\rightarrow$ sum of $*+$ UNDIRECTED GEOGRAPHY $G \in^? \{*, *2\}$
- To win: need to distinguish between $*$ and $*2$.
- PSPACE-hard to determine between values in $\mathcal{N}$.
- Image: BINARY UNDIRECTED GEOGRAPHY[13] Let's play!

[13] http://kyleburke.info/DB/combGames/twoBUG.html

Addition Complexity

- Reduction: GEOGRAPHY→ BINARY UNDIRECTED GEOGRAPHY
- Local vertex and edge transformation + $*$
- 

- 



Closing

- Reasonable arguments against complexity
- CONSTRAINT LOGIC is very useful!
- Sums of shallow partizan games are hard
- Easy winnability does not imply easy values

Thank you!