Prove Your Move

Combinatorial Game Theory for early undergrads with just-in-time Discrete Math

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For my parents, who made everything possible.
-Kyle

Dedicated to my children, who teach me to play.
-Craig

Note: this book was formerly titled "Playing with Discrete Math" and was renamed January 23, 2025.

Useful notation

players	Left/black/blue/Louise, Right/white/red/Richard
$*n$	A nimber, equivalent to a NIM heap of size n
$^*\{ x_1, x_2, \dots, x_k \}$	The options of an impartial game
mex	Smallest excluded non-neg. integer
TT	Tweedledum-Tweedledee Strategy, wherein one player mirrors the other's moves
$z^{\#}$ or $G =^{\#} z$	Equals integer z by counting moves.
$G = \{\mathcal{G}^L \mid \mathcal{G}^R\}$	Game w/ left options $\mathcal{G}^L$ and right options $\mathcal{G}^R$
$G + H$	The *disjunctive sum* of two games; players can choose which game to play on their turn
$*$	$\{ 0 \mid 0 \} = {}^*\{ 0 \}$
$\uparrow$	$\{ 0 \mid * \}$
$+_x$	$\left\{ 0 \,\middle\|\, \{ 0 \mid -x \} \right\}$
$-_x$	$\left\{ \{ x \mid 0 \} \,\middle\|\, 0 \right\} = -\ +_x$
$\Uparrow$	$\uparrow + \uparrow$
$\uparrow*$	$\uparrow + *$
$\mathcal{L}$	Everything > 0 including $\uparrow$, etc.
$\mathcal{R}$	Everything < 0 including $\downarrow$, etc.
$\mathcal{P}$	Next player to move loses, values are zero
$\mathcal{N}$	$\mathcal{N}$ext player to move wins, fuzzy with zero
\/\\|/	A NIM heap
$\mathfrak{n}_S$	A position in SUBTRACTION with set S
$\parallel$	Confused with/incomparable
$\nparallel$	Not confused with
$\lhd\!\mid$	Less than or confused with
$\mid\!\rhd$	Greater than or confused with

Preface

Status of this text

This text is a work in progress. The authors' intention has always been to release it freely to students, faculty, and self-learners. You can download the most recent version from the book's homepage: `http://kyleburke.info/CGTBook.php`. If you are an instructor and would like a copy of the solutions manual in its entirety then please email the authors from your university affiliated email address with your request along with a link to your official university webpage or profile.

Obviously we would be delighted to hear feedback from anyone using this book, either as a learner or an instructor. Errata, organization notes, remarks on the notation we've chosen to use, etc. are all welcome. And instructors, please let us know if you've chosen to use this text in whole or in part for a course you are teaching.

Notes for Instructors

This text has grown (and continues to grow) out of the love the authors share for Combinatorial Game Theory and undergraduate mathematics. While there are myriad texts that focus on Discrete Mathematics, and no dearth of great texts on CGT, we noticed a lack of texts aimed at leading early undergraduate mathematics and computer science students through the world of Combinatorial Games. Discrete Mathematics books tend to focus solely on Discrete Mathematics, while CGT books are usually aimed at more experienced students.

For a Discrete Math Course: Our goal is to teach the fundamental material from a typical early undergraduate course in Discrete Mathematics in the context of Combinatorial Game Theory. In particular, CGT is an excellent framework within which one is exposed to Graph Theory, Inductive Proof, Formal Logic, and nearly every other topic in an introductory course on Discrete Mathematics.

In the table below, we have included what we believe to be the most important topics from such a course (left), which we have connected to CGT topics (right).

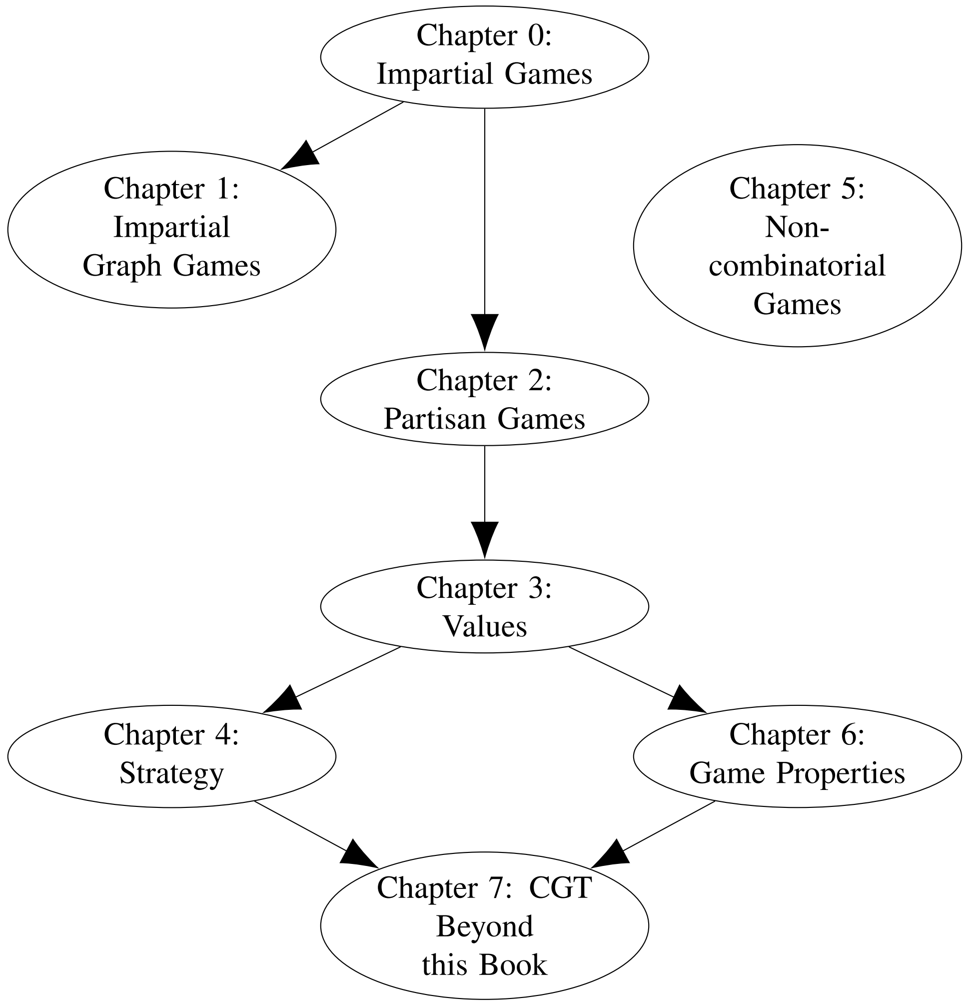
Graphs		Functions/Probability	
Coloring	COL	Bijective	
		Injective	Birthdays
Cartesian product		Surjective	
Connectivity	Game sums		
Digraphs		Recursion	Game values
Paths			
		Discrete probability	Dice
Trees		Expected value	
Bipartite			
Cycles	Graph games		
Eulerian			
Hamiltonian			
Matchings			
Logic		**Set theory**	
Normal form	AVOID TRUE	Cardinality	Birthdays
Truth tables		Countability	
Constructive proof	CHOMP	Cartesian product	Game sums
Connectives	Game trees		
Contradiction	Inequalities	DeMorgan	
Operators	Nim sums	Intersection	
Quantifiers	Nimbers	Set builder	Game trees
		Subtraction	
Direct proofs		Union	
Indirect proofs	throughout		
Inductive proof			
Number theory		**Sequences**	
Binomial coefficient	Cards	Fibonacci	FIBONACCI NIM
Multiplication principle		Beatty	WYTHOFF NIM
Modular arithmetic	Game trees	Arithmetic	
Infinite series	Game values	Characteristic roots	
Infinitesimals	Infinitesimals	Geometric	Game values
		Polynomial	
		Solving recurrences	

In a single semester course for students without a Discrete Mathematics background, we note that most Discrete Mathematics material can be covered by the material that focuses on Impartial Games, Non-combinatorial Games, and Graph Games, while topics like Partisan Games, Values, Strategy, Infinitesimals, etc. can be covered at the instructor's discretion without missing out on important Discrete Mathematics topics.

For a Games Course: This text can be used for a CGT[1] course for students who have already had Discrete Mathematics. In this case, the Math Diversions can simply be skipped, as well as any chapters deemed unnecessary. (This text is designed to be used for both courses, either independently or in sequence.)

Topic Pre-requisites: We also provide a flowchart of pre-requisites between chapters, so advanced readers can pick and choose the CGT topics that interest them the most:

[1] Or any math games.



Examples and Exercises: Throughout this text, we push examples first, believing that this is most helpful for students to learn new concepts. This may seem odd for an intro text in formal mathematics. Aligned with this, we provide answers to many of the exercises in the student version of this text in the appendix. Exercises with answers are marked with a star (⋆), which also acts as a link directly to the solution in the PDF version of this. Each exercise indicates whether the answer is given, so take note when choosing these for assignments.

As we improve this book and add material, we may add new exercises in the middle of the exercise lists. We apologize in advance for any renumbering that this may cause! (Since we're offering this text for free, we're certainly not doing this to purposefully obviate older editions.) We will avoid changing the availability of these answers whenever possible, and be even more averse to revealing previously-hidden answers.

Introduction

Welcome to the wonderful world of Combinatorial Games! We are excited to introduce people to this fascinating subject that combines our love of playing games with the study of discrete math. We hope the use of combinatorial games helps students enjoy the discrete topics here and motivates further study in math, computer science, and maybe even advanced CGT (Combinatorial Game Theory).

Unfortunately, the meaning of the word "game" is a bit ambiguous in Combinatorial Game Theory (CGT). It can refer to any of these things:

- First, it could refer to a *ruleset*, meaning the rules for a game. E.g. "I really enjoy playing the game HEX." In this book, we will study many different rulesets, and we provide a compilation of all of the rules in the appendix, section A.
- Second, it could refer to a game *position*, meaning a state of a game being played. E.g. "What is Blue's best move in this game of CLOBBER?"
- Third, it could refer to the sequence of positions chosen during a game between two opponents. E.g. "Louise and Richard played a game of NIM yesterday." We can instead refer to this as a *game path*.

Because of this ambiguity, it's common for people to avoid using the word "game". In this text, we do the same whenever possible. We will instead use *ruleset*, *position*, or *game path* to differentiate. The reader should understand that in spoken dialogue about CGT, the word game is used all the time, as in the examples above. In most cases the actual case is clear from context, but gamesters[1] are happy to clarify should there be any confusion. Always feel free to ask the question: "What do you mean by the word 'game'?"

[1] *Gamester* is the term for someone who studies CGT.

Contents

0. Impartial Games

> *It don't matter if you're black or white*
>
> \- Michael Jackson, "Black or White"

Let's jump right into a game. We name our two players L and R, after Louise and Richard Guy[1]. Conveniently, these are also the first letters in the words Left and Right, which we will use as pseudonyms for L and R throughout.

SUBTRACTION

SUBTRACTION is a game played on a heap of tokens. Each turn, the current player can remove either 1, 2, or 3 tokens from the pile, provided enough tokens exist. When the pile is empty there are no available moves. In this text, we will use a fancy number script to denote a subtraction position, e.g. $\mathbb{5}$ for a subtraction game with five tokens. Sample game:

$$\mathbb{6} \to \mathbb{5} \to \mathbb{3} \to \mathbb{0}$$

Starting from a pile of 6, the first player takes one, then the second player takes two, then the first player takes all three to win the game.

[1] Richard Guy is one of the three authors of Winning Ways for your Mathematical Plays [1], along with Elwyn Berlekamp and John H. Conway.

Activity: Play SUBTRACTION

Play SUBTRACTION with a friend (or enemy). Whichever player takes the last token is the winner. It's okay to be competitive, but the main objective should be to figure out which player has a winning strategy from different starting amounts of tokens. You may have to play multiple times from the same place to try out different strategies. If you're both okay with it, you can allow backing up instead of starting over completely.
Try playing from starting piles of 5, 10, 15, and 20 tokens. Who has a winning strategy in each of these cases? If you have extra time, try other pile sizes.

The convention of winning used above is known as *normal play*. Other ways to think of this are:

- Last play wins.
- If you can't play, you lose.
- If you don't play, you lose. This accounts for forfeits.

It is possible to use other winning conventions[2] so we won't specify who wins in our ruleset definitions, just when the game ends.

[2]There are many other common conventions! In *misère play*, the last player to play loses instead of wins. Other conventions are less focused on which player plays last. In *scoring play*, the players keep track of a score and the player with the highest score wins at the end. In *maker-breaker play*, one player is trying to attain some goal and the other is trying to prevent it. In this book, we focus only on normal play, but all are valid ways of playing combinatorial games. In many cases, the way we analyze games here doesn't extend to other play conventions, however. New theory is needed; all are active research areas.

Who wins? Who can win?

Throughout this text, it would be burdensome to write "Player X has a winning strategy" or "Can player X win, no matter what their opponent does?" Here are some synonymous sentences:

- X can force a win when going first.
- Assuming they play optimally, X wins going first.
- X has a winning strategy going first.
- No matter what their opponent does, X can win going first.

Instead of using any of these lengthy phrases, we will just write "Player X wins". (Or, in question form, "Does player X win?") Although there may be some scenarios in which a player could lose, we will drop the implicit part and assume optimal play.

You might already see the pattern of first-player vs. second player wins in SUBTRACTION. We would like to develop some more tools that will help us in our study of games.

0.1. Impartial Game Trees

> *I can see clearly now the rain is gone. I can see all obstacles in my way.*
>
> - Johnny Nash, "I Can See Clearly Now"

We would like some tool to show all the possible options from a single position, then include all the moves, or *options*, from those and so on. A natural way to do this uses a tree-like structure, where branches extend downward instead of upward (sorry, Nature).

Each position on the tree is drawn as a *node*. For our *impartial game trees*, we will use only horizontal and vertical lines, so options will be included under positions as a downward-facing pitchfork.

Math Diversion: Graphs, Digraphs, and Trees

A *graph* is a collection of nodes (also called *vertices*) and edges. Usually when we talk about graphs we're concerned with *simple graphs*, where there is at most one edge between a pair of vertices and no edge joins a vertex to itself. Graphs are useful tools for examining pairwise relationships among sets of discrete objects, like friendships, computer architecture, and chemical interactions.

If G is a graph then its *order* $n(G)$ is its number of vertices. The number of edges, $e(G)$, is the *size* of G. If G is connected then between any two vertices u, v of G there is a *shortest path*. This is the smallest collection of successively adjacent vertices from u to v, and the path's *length* is its number of edges. The *distance* from u to v is the length of a shortest path. We denote the set of vertices and edges of G as $V(G)$ and $E(G)$, respectively, and we often write $uv \in E(G)$ or $u \sim v$ to denote that u is adjacent to v.

A *tree* is a connected graph where there's exactly one path between any two nodes. Sometimes we designate a *root*. A vertex with only one neighbor is a *leaf*, while other vertices are called *internal vertices*. An internal vertex has *child* vertices, which are *siblings* of each other, just like in a family tree. A tree with n nodes always has size $(n - 1)$.

The *level* of a node in a rooted tree is its distance from the root. The root is at level 0, its children are at level 1, etc. Nodes in a non-rooted tree have no level.

How many different unrooted trees can you come up with on 6 nodes?

A.: *There are six different trees, including a path on 6 nodes and a star, which is a single center node with the remaining nodes adjacent to it.*

What are the least and greatest possible number of leaves in an unrooted tree of order n?

A: *A tree with n nodes can have as few as two leaves (a path) or as many as $(n-1)$ nodes (a star).*

Sometimes we want to represent not only whether or not a relationship exists between a pair of nodes, but also some hierarchical information about the relationship. For example, if the nodes represent species as in a food web, then we want to orient the edge to show which species preys on the other. An oriented edge is called an *arc*, and a graph with arcs is called a *directed graph* or a *digraph*. If the nodes represent game positions, then we may choose to orient the arcs from positions to their options so that we can follow the arcs from the root to a leaf to represent a game.

Math Diversion: Logical connectives

We can use *logical connectives* to abbreviate mathematical statements. Some of the most common are `and` (denoted $\wedge$), `or` ($\vee$), `not` ($\neg$), and the conditional `if/then` ($\rightarrow$). We can put these together into statements like this:

$$(P \wedge Q) \rightarrow R$$

which we read as "If P and Q are true, then so is R." We can also use them to make claims:

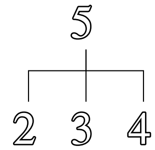
$$\text{For any } x \in \mathbb{R}(\neg(x = 0)) \rightarrow \text{ there is some } y(xy = 1)$$

What does the above statement say? Is it true?

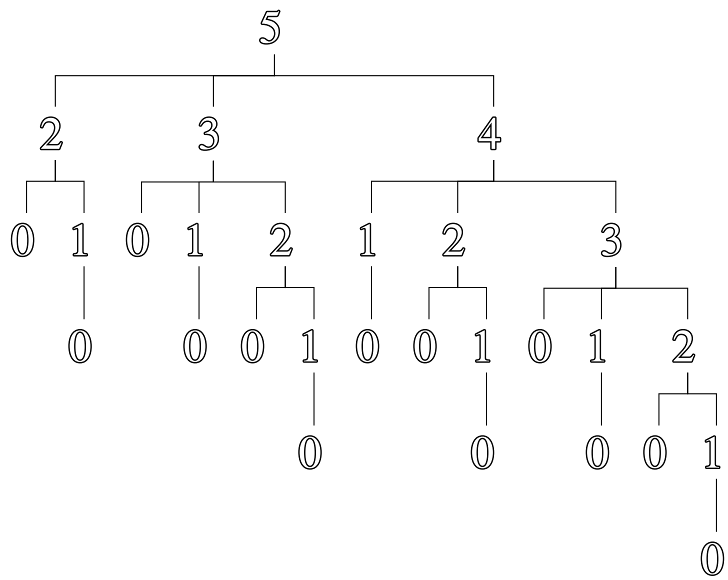
A: *"For any real number x, if x isn't equal to 0 then x has a multiplicative inverse." This is a true statement.*

Write your own example of a true mathematical statement using at least two logical connectives.

For example, this shows the top two levels of an impartial game tree for SUBTRACTION starting with a heap of size 5:



That is only part of the game tree. We can continue it by drawing the options from the positions that are there. If we draw all of the options for each position (all the way down to $\mathbb{0}$) then we get the complete game tree:



Options are new positions that a player can move to directly. In a more general way, *followers* are positions that can be reached after one or more moves. For example, $\mathbb{1}$ is a follower of $\mathbb{5}$, even though it's not an option of $\mathbb{5}$. (All options of a position are also followers of that position.)

Moving from a heap of 1 to 0 is clearly a winning move, as there are no options from $\mathbb{0}$. We could consider coloring (or otherwise marking) the nodes of the tree to mark whether they are wins for the next player or not. E.g. we could mark the zeroes purple and the ones, twos, and threes green. You may already see how we could mark the 4 and 5-nodes. We won't give that away here, but we will show how to do it in Section 0.2.

You probably realized earlier that you could win in SUBTRACTION as long as you didn't have a pile that was a multiple of four. Another way to say this is that losing positions are piles that are *equivalent to zero modulo four*, i.e. that when the pile size is divided by four the remainder is zero. In general, $n \equiv r \pmod{k}$ means that both n and r have the same remainder when divided by k. We can similarly describe our winning positions in this way:

$$n \equiv 1, 2, \text{ or } 3 \pmod{4}$$

which just means that you can win as long as the remainder of n when divided by 4 is $1, 2,$ or 3. This is called *modular arithmetic* and you've probably been using it ever since you learned to tell time. Modular arithmetic comes up often in Combinatorial Game Theory because many of our solutions will be *periodic*, i.e. repeat following a pattern. We will see more of this in later sections.

What if we want to play SUBTRACTION, but we want players to be able to choose from a different set of numbers? We can update our definition to use any numbers by providing a different *set*. We describe that after a quick primer on sets.

Math Diversion: Sets
A *set* is a collection of *elements*. It could be a finite set, like

$$\{4, elbow, ☺\}$$

an infinite set, like

$$\{a, b, aa, ab, ba, bb, aaa, aab, aba, abb, \dots\}$$

or even an empty set $\{\}$, denoted $\emptyset$. A set never has the same element appear more than once, nor does the order of elements matter. So the set $S = \{2, 6, 2, 3\}$ is the same as the set $T = \{6, 3, 2\}$. When we want to say that an element is in a set we use $\in$ or ϵ, so $2 \in T$. We can also look at *subsets* of a set, denoted $A \subseteq B$. These are the sets made up of some, none, or all of the elements in another set. So $\{2, 6\}, \emptyset, T \subseteq T$, among others. Some common sets you may be familiar with are the natural numbers

($\mathbb{N} = \{0, 1, 2, \ldots\}$), integers ($\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$), the real numbers ($\mathbb{R}$), the rationals ($\mathbb{Q}$), and the complex numbers ($\mathbb{C}$).

We can also write sets using *set-builder notation*. The set $\{x \in \mathbb{Z} \mid x > 0\}$ is read "The set of all x in $\mathbb{Z}$ *such that* x is positive." We sometimes abbreviate this set as $\mathbb{Z}^+$ or $\mathbb{Z}_{>0}$.

Can you write out the set of all subsets of S?

A: $\{\{\}, \{2\}, \{3\}, \{6\}, \{2, 3\}, \{2, 6\}, \{3, 6\}, \{2, 3, 6\}\}$

How could you write the set of even integers using set-builder notation?

A: $\{2x | x \in \mathbb{Z}\}$ *or* $\{x \in \mathbb{Z} | x \pmod 2 = 0\}$

There are also some operations we can use on sets to create new sets. If A and B are sets then their *intersection* $A \cap B$ is the set comprised of all elements in both A and B. For example, if E is the set of even natural numbers and T the set of all multiple of three, then $E \cap T$ is the set of all multiples of six. The *union* of A and B, denoted $A \cup B$, is the set of all elements that are in A, or B, or both. So $\{1, 2, 3\} \cup \{2, 4\} = \{1, 2, 3, 4\}$. The *difference* of A and B, $A \setminus B$, is the set of all elements of A that are not in B. Going back to our previous example, $T \setminus E$ is the set of all odd multiple of three. And finally, the *complement* of A, A^c or $\overline{A}$, is the set of absolutely everything not in A.

Let $X = \{1, 2, 3, 4, 5, 6\}$ and $Y = \{2, 4, 6, 8, 10\}$. What are $X \setminus Y, Y \setminus X, X \cup Y$, and $X \cap Y$?

A: $X \setminus Y = \{1, 3, 5\}, Y \setminus X = \{8, 10\}, X \cup Y = \{1, 2, 3, 4, 5, 6, 8, 10\}$, *and* $X \cap Y = \{2, 4, 6\}$

You should convince yourself that

$$(A \cup B)^c = A^c \cap B^c$$

and that

$$(A \cap B)^c = A^c \cup B^c$$

These are well-known as DeMorgan's Laws for sets.

SUBTRACTION

SUBTRACTION is a game played on a heap of n tokens, with a specified set of positive integers ($\mathbb{Z}^+$) known as the subtraction set. Each turn, the current player chooses a number k from the set such that $k \leq n$, and then k tokens are removed from the heap. When n is lower than all elements of the set, there are no more moves and the current player loses. We will describe each position using the fancy number script as before (e.g. $\mathbb{6}$) or by also including the subtraction set below if it's not understood from context (e.g. $\underset{\{1,5,6\}}{\mathbb{6}}$).

$$\underset{\{1,2,4\}}{\mathbb{9}} \to \mathbb{7} \to \mathbb{3} \to \mathbb{1} \to \mathbb{0}$$

The first player takes two tokens to move to a heap of 7. The second player then takes four tokens to move to a heap of size 3. Taking three is not in the subtraction set, so the first player instead takes two, and the second player responds by taking the final token to win.

We can refer to the version we played in the Introduction as SUBTRACTION-$\{1,2,3\}$.

Let's see how fun this game is when played with a different set!

Activity: Play (updated) SUBTRACTION

Play SUBTRACTION-$\{1,2,4\}$ with a friend . Again, the main objective for you both should be to figure out who wins from different starting amounts of tokens.

Try playing starting with 5, 10, 15, and 20 tokens. Who wins in each of these cases? Is it easier or harder to figure out who wins than in the version with the set $\{1,2,3\}$?

If you are enjoying this, try playing with another set you think will be "easy" and another set you think will be "hard". Were you and your friend correct in your hypotheses?

Exercises for 0.1

⋆ **0)** $S = \{x \in \mathbb{N} \mid x > 5 \text{ and } x < 10\}$. Rewrite S without using set-builder notation. (Answer 0.1.0 in Appendix)

⋆ **1)** $S = \{2k + 1 \mid k \in \mathbb{N} \cup \{0\} \text{ and } k \leq 6\}$. Rewrite S without using set-builder notation. (Answer 0.1.1 in Appendix)

2) $S = \{x \in \mathbb{Z} \mid |x| < 4\}$. Rewrite S without using set-builder notation.

3) $S = \{b + 1 \mid b \in \mathbb{Z} \text{ and } b < 5 \text{ and } b > 10\}$. Rewrite S without using set-builder notation.

4) $S = \{2k \mid k \in \mathbb{Z} \text{ and } \left|k\right| \leq 3\}$. Rewrite S without using set-builder notation.

5) $S = \{x \in \mathbb{N} \mid x > 3 \text{ and } x < 5\}$. Rewrite S without using set-builder notation.

⋆ **6)** Draw the first two levels of the impartial game tree from $\underset{\{1,2,3\}}{\mathbb{3}}$. (Your diagram should show the initial position and all the options from that position.) (Answer 0.1.6 in Appendix)

7) Draw the first two levels of the impartial game tree from $\underset{\{1,2,3\}}{\mathbb{2}}$. (Your diagram should show the initial position and all the options from that position.)

⋆ **8)** Draw the first two levels of the impartial game tree from $\underset{\{1,2,3\}}{\mathbb{4}}$.(Your diagram should show the initial position and all the options from that position.) (Answer 0.1.8 in Appendix)

9) Draw the first two levels of the impartial game tree from $\underset{\{1,3,4\}}{\mathbb{6}}$. (Your diagram should show the initial position and all the options from that position.)

⋆ **10)** Draw the entire impartial game tree for $\underset{\{1,2,3\}}{\mathbb{2}}$. Is there a winning move for the first player? Justify your answer. (This is a continuation of 0.1.7 .) (Answer 0.1.10 in Appendix)

11) Draw the (impartial) game tree for $\underset{\{1,2,3\}}{3}$. Is there a winning move for the first player? Justify your answer. (This is a continuation of exercise 0.1.6.)

⋆ **12)** Draw the impartial game tree for $\underset{\{1,2,3\}}{4}$. Which of the two players (first or second) has a winning strategy from the pile of 4? Justify your answer. (This is a continuation of exercise 0.1.8.) (Answer 0.1.12 in Appendix)

13) Draw the (impartial) game tree for $\underset{\{2,3\}}{4}$. Which of the two players (first or second) has a winning strategy from the pile of 4? Justify your answer.

⋆ **14)** What are all of the followers of $\underset{\{1,3\}}{3}$? (Answer 0.1.14 in Appendix)

15) What are all of the followers of $\underset{\{2,3\}}{5}$?

16) What are the losing positions (for the first player) when we play SUBTRACTION-$\{2,4,6\}$? Write your answer using modular arithmetic.

⋆ **17)** If $x \equiv 3 \pmod 5$ and $y \equiv 1 \pmod 5$, then what is $x + y \pmod 5$? (Answer 0.1.17 in Appendix)

18) If $x \equiv 3 \pmod 5$, $y \equiv 3 \pmod 5$, and $z \equiv 3 \pmod 5$, then what is $x + y + z \pmod 5$?

⋆ **19)** How do a natural number and its square compare under arithmetic $\pmod 2$? i.e. are n and n^2 always equivalent $\pmod 2$, always non-equivalent $\pmod 2$, or sometimes equivalent and sometimes not? (Answer 0.1.19 in Appendix)

20) How do a natural number and its square compare under arithmetic $\pmod 3$? i.e. are n and n^2 always equivalent $\pmod 3$, always non-equivalent $\pmod 3$, or sometimes equivalent and sometimes not?

0.2. Impartial Outcome Classes

Activity: Subtraction Winnability Table

Play SUBTRACTION-$\{1,2,6\}$ with a friend. See if together you can figure out which player wins from each starting pile size. Copy down this table, check the first entries, then fill in the missing ones. (If this is an in-class activity, how many extra entries can you get to before your teacher asks everyone to stop?)

Size	0	1	2	3	4	5	6	7	8	…
Winner	2^{nd} player	1^{st}	1^{st}	2^{nd}	1^{st}	?	?	?	?	…

For any position, exactly one of the two players has a winning strategy: either the first player (the next one to play) or the second player (the previous one). "First" and "Second Player" are often used in an all-encompassing way (e.g. "the first player to take a turn this game.") instead of relative to the current game position. Because of this, in CGT, we instead say "$\mathcal{N}$ext" and "$\mathcal{P}$revious" to refer to the two players based on who is currently considering their turn. Then we define two sets:

- $\mathcal{P}$: $\{G \mid \mathcal{P}$revious player wins G $\}$
- $\mathcal{N}$: $\{G \mid \mathcal{N}$ext player wins $G\}$

For example, $\underset{\{1,2,3\}}{3} \in \mathcal{N}$, while $\underset{\{1,2,3\}}{4} \in \mathcal{P}$.

We refer to these sets as *outcome classes*. It is also common to refer to games in $\mathcal{P}$ as $\mathcal{P}$-positions, and $\mathcal{N}$ games as $\mathcal{N}$-positions. We can rephrase the table from the beginning of this section to use this new notation:

Size	0	1	2	3	4	5	6	7	8	9	…
Outcome class	$\mathcal{P}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{P}$	$\mathcal{N}$	?	?	?	?	?	…

We can determine the outcome class of an impartial game position with the following recursive rule:

$$G \in \begin{cases} \mathcal{P}, & \text{All of } G\text{'s options are in } \mathcal{N} \\ \mathcal{N}, & G \text{ has an option in } \mathcal{P} \end{cases}$$

It might be easier to think of it in this (equivalent) way:

$$G \in \begin{cases} \mathcal{P}, & G \text{ does not have an option in } \mathcal{P} \\ \mathcal{N}, & G \text{ has an option in } \mathcal{P} \end{cases}$$

This means we can use a game tree to find the outcome class of each node!

Math Diversion: Recursion

There are a number of ways to get one mathematical result from another. One method is called *recursion*. A relationship that is defined by recursion is called a *recurrence relation*. For example, we can define a function that takes a natural number and multiplies it by the value of the function on the previous natural number, written like this

$$f(n) = \begin{cases} 2 & n = 0 \\ n \cdot f(n-1) & \text{otherwise} \end{cases}$$

We have just defined a recursive function. Notice that we also have to state what happens when $n = 0$. This is called the *base case* of the recurrence relation. Otherwise it's impossible to know what $f(1)$, $f(2)$, or any other value is.

Take a minute to determine what $f(5)$ is.

A.: $f(0) = 2$, $f(1) = 1 \cdot 2 = 2$, $f(2) = 2 \cdot 2 = 4$, $f(3) = 3 \cdot 4 = 12$, $f(4) = 4 \cdot 12 = 48$, $f(5) = 5 \cdot 48 = 240$

Our outcome class labels are also recursively defined. Each node's label in a rooted tree is determined by the labels of its children. We've also established that a node without any children is assigned the label $\mathcal{P}$.

As another example, let's define a function g on $\mathbb{N}$ by

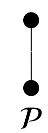
$$g(n) = \begin{cases} 1 & n = 0 \text{ or } 1 \\ g(n-1) + g(n-2) & \text{otherwise} \end{cases}$$

What is $g(5)$?

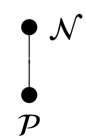
A.: $g(0) = 1, g(1) = 1, g(2) = 1 + 1 = 2, g(3) = 2 + 1 = 3, g(4) = 3 + 2 = 5$, *and so* $g(5) = 5 + 3 = 8$. *This may be familiar to you as the famous Fibonacci Sequence.*

Let's look at an extremely simple impartial game tree and label each node with its outcome class.

All nodes with no children have no options, so they don't have any options in $\mathcal{P}$. That means they themselves are in $\mathcal{P}$. Let's label that node.

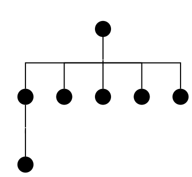


Then, since that top node has an option in $\mathcal{P}$, it falls into the $\mathcal{N}$ category.

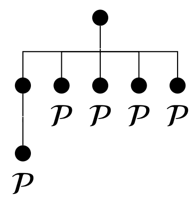


This is very useful. We can determine the outcome class of a game just by looking at its game tree! If we draw the tree, we don't even need to know how the rules otherwise work.

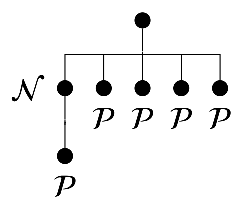
Let's do one more example:



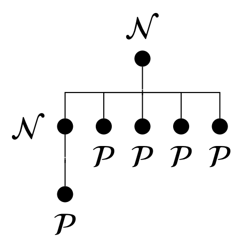
All nodes with no children become $\mathcal{P}$:



Then, working upwards, the remaining node on the second level is in $\mathcal{N}$:



Finally, we can also label the top $\mathcal{N}$, because it has (at least) one $\mathcal{P}$ option:



Computational Corner: Running Time
When analyzing the speed of a function, we can't give an exact running time (e.g. in microseconds) because that will change with respect to the size of the function's inputs. We also can't express it as a function of that size in a specific unit of time (e.g. $9n$ microseconds) because that will change based on the hardware and operating system we run the code on.

Instead, we express the running time as a function of the number of steps it requires in the worst case on a given size. Then, when we use Big-O notation to classify running times, we drop the coefficients of terms as well as non-dominant terms.[a]

For example, this function finds the maximum element in a list of numbers:

```
def max(numbers):
    maximum = numbers[0]
    i = 0
    while i < len(numbers):
        number = numbers[i]
        if number > maximum:
            maximum = number
        i += 1
    return maximum
```

In the worst case (each element is a new maximum) `max` uses $6n+5$ steps in its execution, where n is the length of `numbers`[b]. Using Big-O notation, we classify this as a $O(n)$, or linear, function. This tells us that the running time increases linearly with the length of the list.

In general, we will not go so far as to find the actual number of steps, then classify with Big-O. Instead, we will skip to Big-O by considering specific instructions inside of loops. In the `max` example above, we can count the comparisons inside the loop. The check `if number > maximum` is run once every time through the loop. There are no other instructions deeper in the function that happen more often. There are also no other functions (aside from `len`) in the loop or elsewhere in the function that might have a more complicated running time on their own.

[a] We are glossing over a ton of the theory behind this practice. See https://en.wikipedia.org/wiki/Computational_complexity_theory or any text on algorithm analysis for more information. We are also choosing to use Big-O instead of Big-Θ, though the latter is arguably more appropriate.

[b] You can verify this calculation in exercise 0.2.9 . We count the execution of `len` as only taking one step, though it certainly takes more.

Computational Corner: Python Outcome Classes

Let's consider determining outcome classes in Python 3. If we are programming quickly, we can model each outcome class as a string with a single character, either `'N'` or `'P'`. Then, given a list of these, we could write a function to determine the overall outcome class that could be used like this:

```
>>> out = get_parent_impartial_outcome_class(['N', 'P', 'N'])
>>> out
'N'
>>> out = get_parent_impartial_outcome_class(['N', 'N', 'N',
'N'])
>>> out
'P'
>>> out = get_parent_impartial_outcome_class([])
>>> out
'P'
```

Here is an example of how we could write that function:[a]

```
def get_parent_impartial_outcome_class(option_classes):
    expected_class = 'P'
    for option in option_classes:
        if option == 'P':
            expected_class = 'N'
    return expected_class
```

This function's loop requires it to check every option. We shouldn't have to do that, however! If we see a $\mathcal{P}$ option, we can cut out of the function and return $\mathcal{N}$. Here's how we can rewrite the function to do that:

```
def get_parent_impartial_outcome_class(option_classes):
    expected_class = 'P'
    for option in option_classes:
        if option == 'P':
            return 'N'
    return expected_class
```

And, then, since we are never changing `expected_class`, we can drop that variable:

```
def get_parent_impartial_outcome_class(option_classes):
    for option in option_classes:
        if option == 'P':
            return 'N'
    return 'P'
```

[a] We use the variable `option` instead of `class` because `class` is a keyword in Python.

Computational Corner: Recursive Functions
Instead of having an 'N' or 'P', we could recursively have another list that represents those options. That list could contain either characters for the outcome classes ('N' and 'P', as we've been doing) and other lists. For example, what if we were asked to evaluate the list ['N', ['N', 'N'], ['P']]?

We can modify our previous code to be recursive. If we see a list, we'll make a recursive call.

```
def get_parent_impartial_outcome_class(options):
    for option in options:
        if isinstance(option, list):
            option_class = get_parent_impartial_outcome_class(option)
            option = option_class
        if option == 'P':
            return 'N'
    return 'P'
```

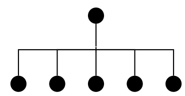
Notice that this doesn't fit the normal basic structure of a fruitful recursive function:

```
if base_case:
    return base_case_value
else:
    #maybe do some work
    result = recursive_call()
    #maybe do some work
    return something
```

This is because we may be making a bunch of recursive calls as we have to iterate through the entire list.

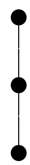
Exercises for 0.2

⋆ **0)** What is the outcome class at the root of this impartial game tree?



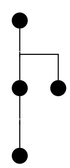
Justify your answer by labeling the nodes of the tree. (Answer 0.2.0 in Appendix)

1) What is the outcome class at the root of this impartial game tree?



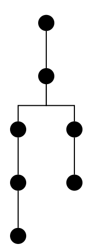
Justify your answer by labeling each node of the tree with its outcome class.

⋆ **2)** What is the outcome class at the root of this impartial game tree?



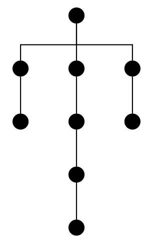
Justify your answer by labeling each node of the tree with its outcome class. (Answer 0.2.2 in Appendix)

3) What is the outcome class at the root of this impartial game tree?



Justify your answer by labeling each node of the tree with its outcome class.

4) What is the outcome class at the root of this impartial game tree?

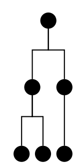


Justify your answer by labeling each node of the tree with its outcome class.

⋆ **5)** Let $G = \underset{\{1,2,3\}}{3}$. What is the $o(G)$? Justify your answer by drawing the game tree and labeling each node with its outcome class. (This is a continuation of exercise 0.1.11.) (Answer 0.2.5 in Appendix)

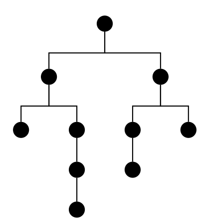
6) Let $G = \underset{\{1,2,3\}}{4}$. What is $o(G)$? Justify your answer by drawing the game tree and labeling each node with its outcome class. (This is a continuation of exercise 0.1.12.)

⋆ **7)** Find the outcome class of this tree:



Then add one new child to one of the leaves to flip the outcome class at the root of the tree. (Yes, you need to show the work to derive the outcome class of the new tree.) (Answer 0.2.7 in Appendix)

8) Find the outcome class of this tree:



Then add one new child to one of the leaves to flip the outcome class at the root of the tree. (Yes, you need to show the work to derive the outcome class of the new tree.)

⋆ **9)** Verify that the `max` function in this chapter requires $6n+5$ Python instructions to run. Assume that the `len` function takes one step to complete. (Answer 0.2.9 in Appendix)

10) Consider this (incorrect) version of our function to determine a position's outcome class.

```
def get_parent_impartial_outcome_class(option_classes):
    expected_class = 'P'
    for option in option_classes:
        if option == 'P':
            expected_class = 'N'
        elif option == 'N':
            expected_class = 'P'
    return expected_class
```

Find a list where this returns the wrong thing and explain why that happens. Fix this by replacing one line with the Python command `pass`, which does nothing.

11) In the practice of Defensive Programming, it's a good idea to check for unexpected values and throw an error if you see them. Consider this version of our function from before:

```
def get_parent_impartial_outcome_class(option_classes):
    expected_class = 'P'
    for option in option_classes:
        if option == 'P':
            return 'N'
        elif option != 'N':
            ... your code here...
    return expected_class
```

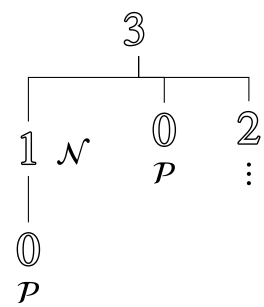
Finish this by raising an exception in the empty branch.[3]

[3]More information about errors can be found here: `https://docs.python.org/3/tutorial/errors.html`.

0.3. Trimming Game Trees

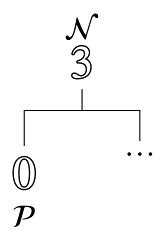
When drawing out an impartial game tree, there are often some reasonable shortcuts we can employ to avoid analyzing the entire tree.

First, when we're determining the outcome class for a position, we are only concerned with whether each position has a $\mathcal{P}$-option or not. If we find one, then we don't need to continue evaluating the other options. For example, consider this incomplete evaluation of $\underset{\{1,2,3\}}{\mathbb{3}}$:



Even though the game tree for $\mathbb{2}$ isn't fully drawn out and evaluated, we still know the outcome class of $\mathbb{3}$ because we found an option in $\mathcal{P}$. This *trimmed game tree* still proves that $\mathbb{3}$ is in $\mathcal{N}$.

In the above example, there was only one option of $\mathbb{3}$ left after finding the $\mathcal{P}$-option, but if we found it sooner, then we wouldn't need to analyze any of the remaining options. In that case, we can note that there are multiple options left unexplored by using a horizontal ellipse (dots) instead of a vertical one:

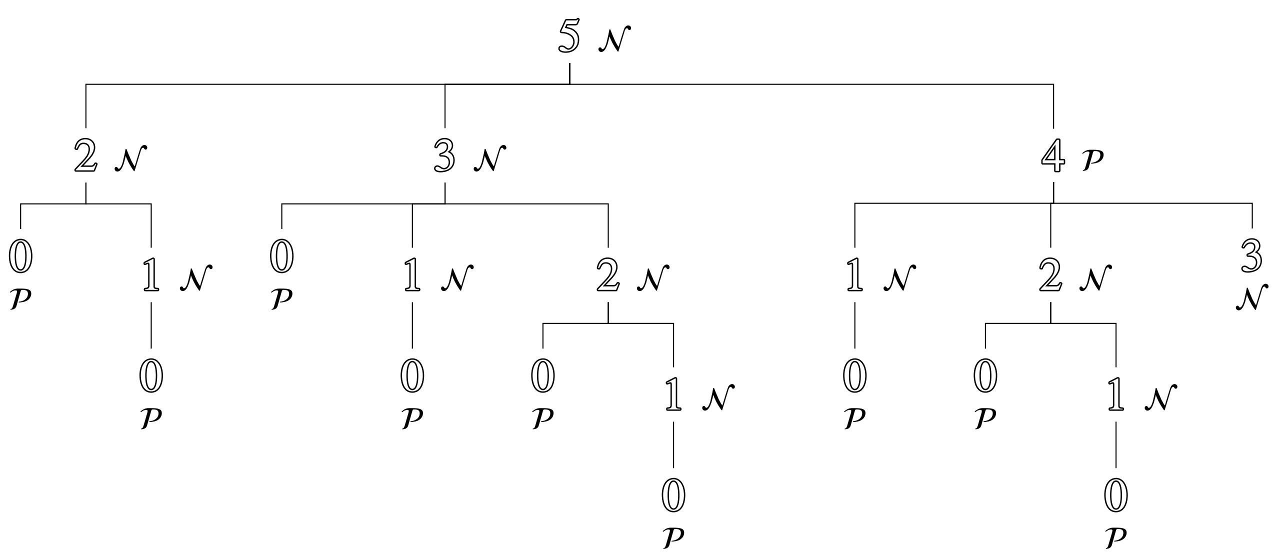


Some important notes about this shortcut:

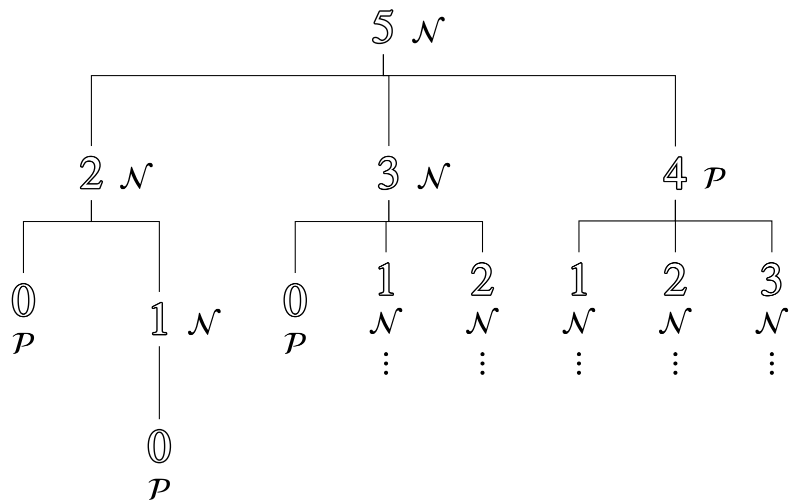
- First, we can't use this trick for $\mathcal{P}$-positions. We still need to evaluate all of their options to show that none of them are $\mathcal{P}$ themselves.

- Second, this trick is only useful for determining outcome classes. In later sections, when we are finding game *values*, then we can't stop just because we see a specific option.

For the second shortcut, if we have drawn out the subtree for a position elsewhere in the whole tree, we don't need to draw it again. For example, consider this game tree for $\mathbb{5}$:



The tree for $\mathbb{3}$ isn't derived on the far right side. Since it was already done out in the table, we know it's already in $\mathcal{N}$ without repeating the work. Notice that we can simplify this further by doing the same with $\mathbb{2}$ and $\mathbb{1}$.



This is a legitimate truncation of the tree. Even though we've removed so much of the original tree, it is still a valid diagram proving that $5 \in \mathcal{N}$.

Exercises for 0.3

⋆ **0)** Use a trimmed game tree to find and show the outcome class of $6_{\{1,3,4\}}$. (This is a continuation of exercise 0.1.9.) What is the smallest tree you can draw that proves your result? (Answer 0.3.0 in Appendix)

1) Use a trimmed game tree to find and show the outcome class of $6_{\{1,5\}}$.

⋆ **2)** Use a trimmed game tree to find and show the outcome class of $5_{\{1,3\}}$. (Answer 0.3.2 in Appendix)

3) Use a trimmed game tree to find and show the outcome class of $5_{\{2,3\}}$.

⋆ **4)** Use a trimmed game tree to find and show the outcome class of $9_{\{2,3\}}$. (Answer 0.3.4 in Appendix)

5) Use a trimmed game tree to find and show the outcome class of $\underset{\{2,3\}}{10}$.

6) Use a trimmed game tree to find and show the outcome class of $\underset{\{2,3\}}{11}$.

⋆ **7)** Use a trimmed game tree to find and show the outcome class of $\underset{\{2,4,5\}}{9}$. (Answer 0.3.7 in Appendix)

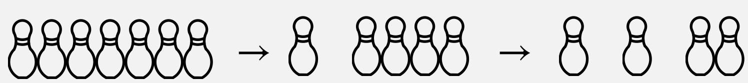
8) Use a trimmed game tree to find and show the outcome class of $\underset{\{2,4,5\}}{14}$.

0.4. Game Sums

Let's look at the ruleset for a new game.

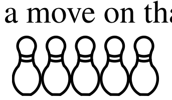
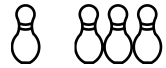
KAYLES

KAYLES is a bowling game created by Henry Dudeney in 1908[], derived from the lawn game Skittles. Each turn, players bowl a ball towards a row of bowling pins that may include some gaps. The ball can either knock over (remove) a single pin or two adjacent pins. Removed pins leave gaps in the row. The game ends when all pins have been removed.

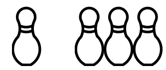
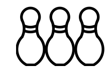


The first bowl takes out the second and third pins, the second bowl removes only the fifth pin.

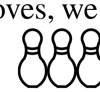
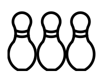
In the sample game, the moves bowling in the middle separate the position into two separate components. These two parts are independent of each other; on each player's turn, they pick one of the components and make a move on that side.

As another example, starting with a row of five pins,  , bowling down the second pin results in  . In this new position, the sections are independent of each other. Although the moves a player makes on one side may affect the moves the other player will (or should) respond with, those moves will not affect the state of the other side directly.

We can use this idea of independent components as the basis for an addition operator on positions. For this "game sum" [4], we use the familiar + operator. For any two positions, G and H, $G + H$ is a game position where the current player picks either G or H (not both) and makes a move in one of them[5]. The result of that move is the sum of the resulting component from the move and the entire original other position. That means that if a player chooses the option, J, of H, the resulting overall game is $G + J$. If they instead were to choose an option, say F, of G, then the resulting overall game would be $F + H$.

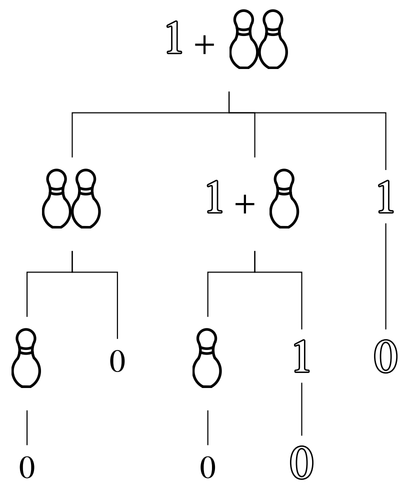
Continuing with our previous example,  =  +  , playing on the left side results in removing that single pin and leaving the three on the right. Playing instead on the right results in the one pin on the left added to whatever that move on the right side leaves.

We are not restricted to considering game sums between positions in the same ruleset! What if we add $\mathbb{1}_{\{1,2,3\}}$ to  ? Let's check out the game tree. But, before we do, here are two rules about how we will treat completed positions (with no moves):

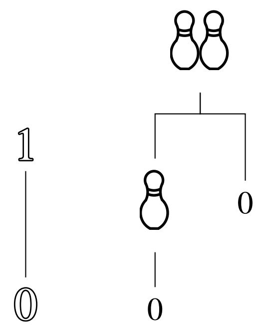
- First, when a position occuring as part of a sum has no more moves, we will just drop that component. For example, we will simplify $\mathbb{0}$ +  as  .
- Second, some rulesets don't have a great visual representation for positions with no options. We will represent these with a zero: 0. For example, the only move on  is to 0.

[4]More formally, "disjunctive (game) sum". There are other ways to define sums on games, so the word disjunctive is usually only used in the context of other sums.
[5]We will see a more formal definiton of game sums in section 2.1.

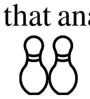
Here's the game tree:

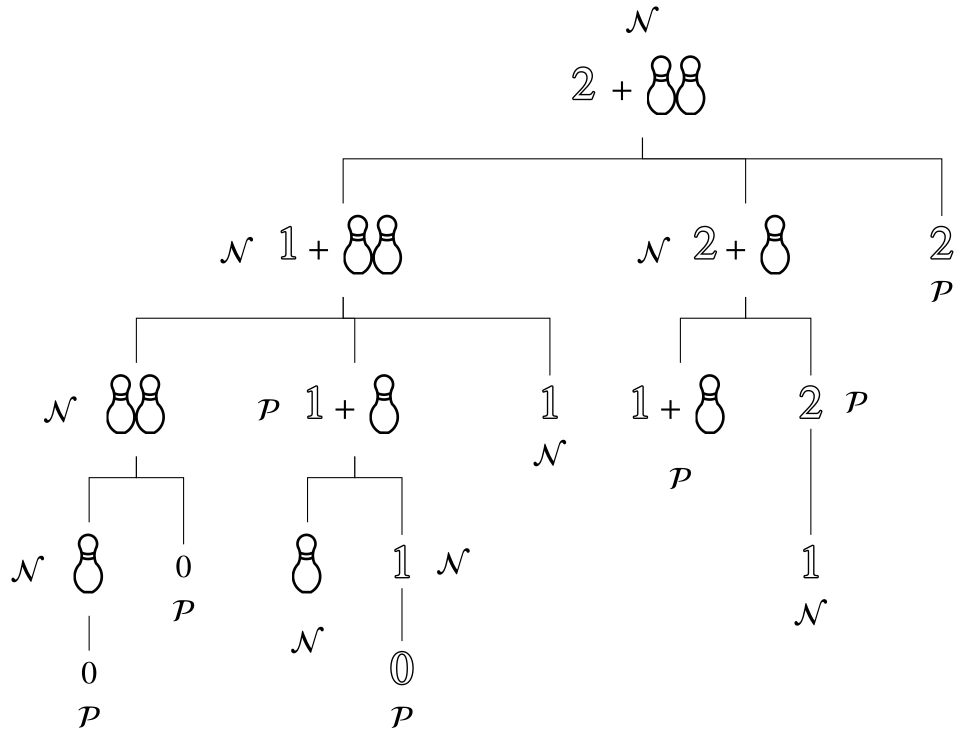


This tree is larger than the trees for the individual components:



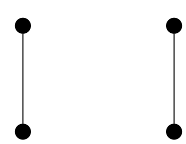
This explosion in size is extremely important. The focus of much of this book is learning how to analyze the components individually rather than the sum all at once. If we can do that, then we're using a sort of divide-and-conquer strategy: break apart a position into its independent components, analyze those parts, then use that analysis to reason about the whole position.

We can already do some of that analysis with the tools we have so far. Consider the positions $\mathbb{2}_{\{1,3\}}$ (in $\mathcal{P}$) and  (in $\mathcal{N}$). What is the outcome class of their sum? Let's use a game tree:

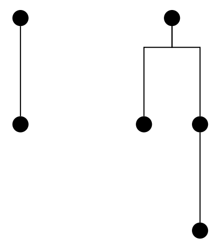


The sum is in $\mathcal{N}$. It turns out that whenever we add a $\mathcal{P}$-position to an $\mathcal{N}$-position, the result is always an $\mathcal{N}$-position! In the same way, if we add two $\mathcal{P}$-positions, the sum is another $\mathcal{P}$-position (see this section's exercises). $\mathcal{P}$-positions always act like the number zero in sums of games. For that reason, the official name of $\mathcal{P}$ is "Zero".

$\mathcal{N}$ doesn't have quite the same effect. The sum of two $\mathcal{N}$-positions could be either $\mathcal{N}$ or $\mathcal{P}$. For example, the sum of these two $\mathcal{N}$ trees is $\mathcal{P}$:



However, the sum of these $\mathcal{N}$ trees is still $\mathcal{N}$:



For this reason, $\mathcal{N}$ is named "Fuzzy".

	$G + H$	H: $\mathcal{P}$	H: $\mathcal{N}$
G	$\mathcal{P}$	$\mathcal{P}$	$\mathcal{N}$
	$\mathcal{N}$	$\mathcal{N}$	?

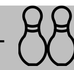
Thus far we have explicitly used trees to represent options in a game. However, there is no reason to restrict ourselves in this way. Notice above that there are multiple nodes representing the position $\mathbb{1}$ +  . The subtrees that follow these nodes are identical. We can combine these nodes into a single node and repeat the process for every position that is repeated in the tree. What results is a *game graph*, and it usually requires fewer nodes than a game tree.

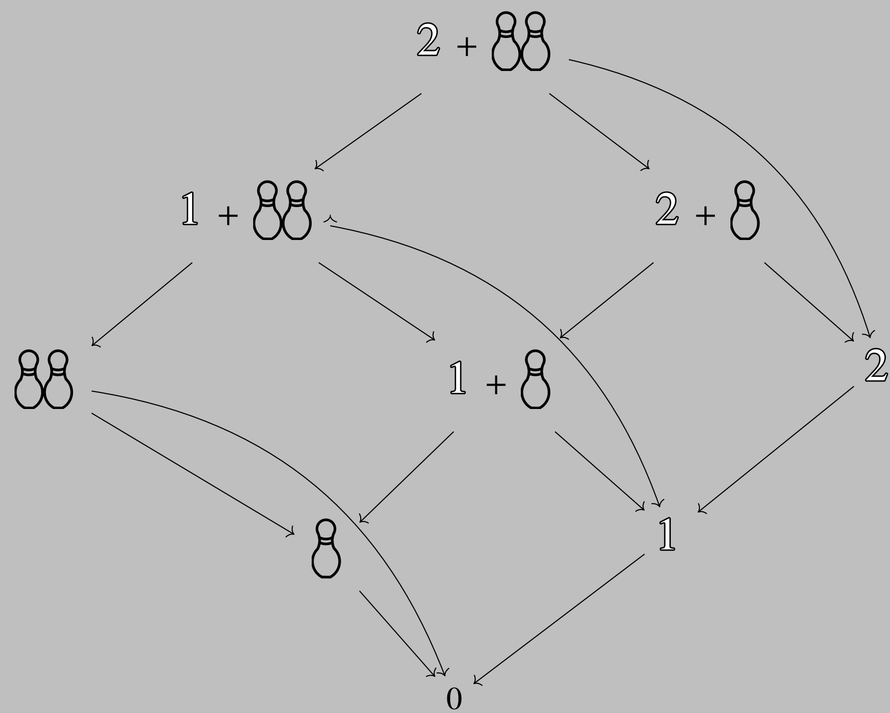
Math Diversion: Graph products
The *Cartesian product* (named for René Descartes) of two sets A and B is defined as

$$A \times B = \{(a, b) \mid a \in A, b \in B\}$$

That is, the collection of ordered pairs where the first term comes from A and the second term from B. Cartesian coordinates in the plane, with which you're already familiar, are elements of $\mathbb{R} \times \mathbb{R}$. You may also have seen this represented as $\mathbb{R}^2$.

We can also take the Cartesian product of a pair of graphs. Let G and H be graphs. The Cartesian product of G and H (sometimes just called the product) is denoted $G \square H$. Its vertices are the ordered pairs $V(G) \times V(H)$, and it has edges between (g_1, h_1) and (g_2, h_2) whenever $g_1 = g_2$ and $h_1 \sim h_2$, or $g_1 \sim g_2$ and $h_1 = h_2$.

Below is the Cartesian product of the tree for $2_{\{1\}}$ +  . (For those who are already familiar with partisan games, the direction of the arrows does not indicate that those moves are only available for either L or R.):

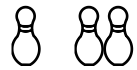


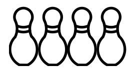
If G is an edge adjoining two vertices, what does the product $G\square G$ look like? What about $(G\square G)\square G$? Can you generalize it to any number of copies of G?

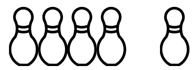
A.: *$G\square G$ is a square, also called a 4-cycle and denoted C_4. $(G\square G)\square G$ is a cube. As we increase the number of factors in our product we increase the dimensions of our cube.*

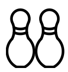
If G_1, G_2 are games then the graph of the game sum G_1+G_2 is the Cartesian product of the game graphs of G_1 and G_2. You may want to verify this for yourself, but only with very small games.

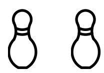
Exercises for 0.4

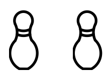
⋆ **0)** Use a (partial) game tree to find the outcome class of  . (Answer 0.4.0 in Appendix)

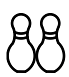
1) Use a (partial) game tree to find the outcome class of  .

2) Use a (partial) game tree to find the outcome class of  .

⋆ **3)** Find the outcome class of $3_{\{1,2\}}$ +  . If possible, find and use the outcome classes for the two components and use that to justify your answer. If that fails, draw the game tree for the sum. (Answer 0.4.3 in Appendix)

4) Find the outcome class of $3_{\{1,2\}}$ +  . If possible, find and use the outcome classes for the two components and use that to justify your answer. If that fails, draw the game tree for the sum.

⋆ **5)** Find the outcome class of $3_{\{1,2,3\}}$ +  . If possible, find and use the outcome classes for the two components and use that to justify your answer. If that fails, draw the game tree for the sum. (Answer 0.4.5 in Appendix)

6) Find the outcome class of $3_{\{1,2,3\}}$ +  . If possible, find and use the outcome classes for the two components and use that to justify your answer. If that fails, draw the game tree for the sum.

7) Let $A = \{a, b, c\}$ and $B = \{x, y, z\}$. How many elements are in $A \times B$?

8) Find $P_3 \square P_3$, where P_3 is a path graph on three vertices.

9) Find $P_3 \square P_3 \square P_3$, where P_3 is a path graph on three vertices.

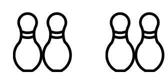
⋆ **10)** Prove that if games G and H are both in $\mathcal{P}$, then the game $G + H$ is in $\mathcal{P}$. (Answer 0.4.10 in Appendix)

11) Prove that if the game G is in $\mathcal{P}$ and the game H is in $\mathcal{N}$, then the game $G + H$ is in $\mathcal{N}$.

0.5. Tweedledum and Tweedledee

"What's good for you is good for me"
Says Tweedle-dee Dum to Tweedle-dee Dee

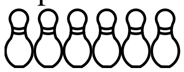
- Bob Dylan[6], "Tweedle Dee & Tweedle Dum"'

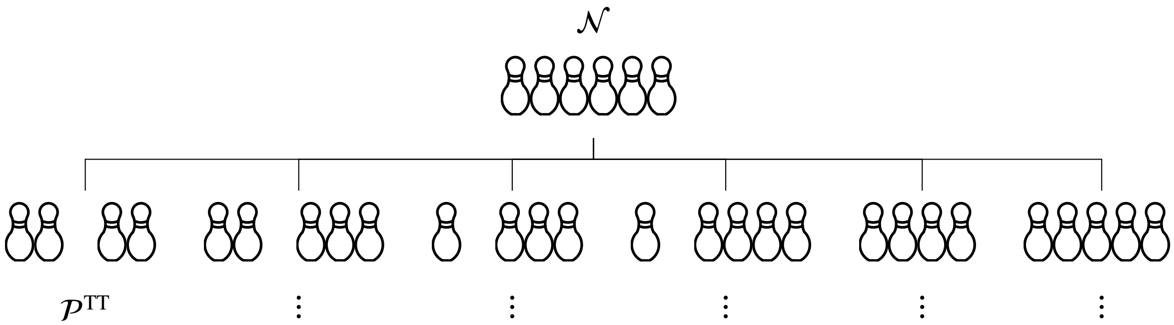
You may have already noticed that positions like  and  are in $\mathcal{P}$ without having to draw out the game tree. There is a common high-level strategy for the previous player to win on any position that is a sum of two equal impartial components called *Tweedledum and Tweedledee*[7].

The strategy is simple to explain and does not require any of the theory we've built up so far. Whatever the next player does to one side of the sum, the previous player will do that to the other side. After both moves have been made, the current position will either be terminal, or the game will still be the sum of two equal positions. Thus, the previous player can continue this strategy until they reach a terminal position. The next player can never be the one to reach the terminal position because they can only make a move on one of the components.

In general, this means that for any impartial position G, $G + G \in \mathcal{P}$. We don't even need to determine the outcome class of G. As you can imagine, this

[7] This is a reference to the two characters in Lewis Carroll's Through the Looking Glass

simplifies evaluation of many positions. For example, it is very easy to see why  is an $\mathcal{N}$-position:

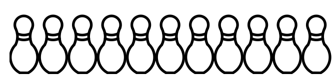


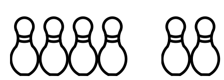
Some important notes for using this.

- First, the $\mathcal{P}^{\text{TT}}$ above means we know that this is a $\mathcal{P}$-position, because the previous player can follow the Tweedledum-Tweedledee strategy (TT).[8]
- Second, in section 2.4, we will generalize this principle so we can use it in more cases!

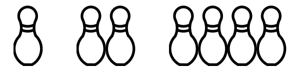
"Ditto," said Tweedledum.
"Ditto, ditto!" cried Tweedledee.[9]

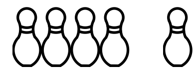
Exercises for 0.5

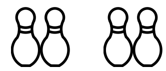
⋆ **0)** Find and prove the outcome class of , a KAYLES row of 11 pins. (Answer 0.5.0 in Appendix)

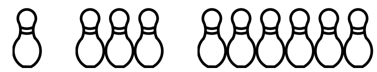
1) Find and prove the outcome class of .

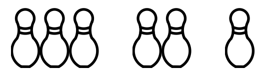
[8]Beware: this is not standard notation outside of this text.
[9]From *Through the Looking Glass* by Lewis Carroll.

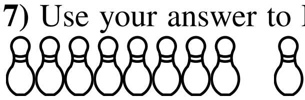
⋆ **2)** Find and prove the outcome class of  . (Answer 0.5.2 in Appendix)

3) Find and prove the outcome class of  .

⋆ **4)** Find and prove the outcome class of  + $\mathbb{3}_{\{1,3\}}$. (Answer 0.5.4 in Appendix)

5) Find and prove the outcome class of  .

⋆ **6)** Find and prove the outcome class of  . (Answer 0.5.6 in Appendix)

7) Use your answer to Exercise 0.5.6 to find and prove the outcome class of  .

0.6. Nimbers

> *But Zero doesn't care what the stats add to*
> *'Cause winning is all zero ever wanted to do*
>
> - Phonte, "Why not 0"'

Finding the outcome classes of two components can help determine the outcome class of their sum, but only if at least one of them is $\mathcal{P}$. If both are $\mathcal{N}$, then we either need to trace out the whole game tree or find another solution. Thankfully, we have that other solution: we can find the *values* of the individual components and use those together to determine winnability!

For impartial games, these values are called *nimbers*. Before we learn about them specifically, let's learn about the ruleset they're named after, NIM.

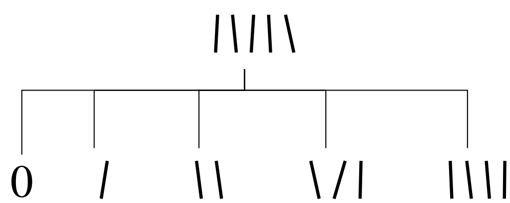
NIM

NIM is an ancient game that has been played in different forms across different cultures. In 1901, Charles Bouton coined the name "Nim" and described the complete theory[2].
The game play consists of removing objects (classically tokens, sticks, or pebbles) from piles ("heaps" or "Nim heaps"). Each turn, a player removes objects (at least one) from exactly one heap. The game ends when all heaps are empty.

| /\ /\/ → / \\ → / |

First move takes the entire last heap; the second takes one stick from the second.

Each NIM heap is essentially a SUBTRACTION pile of the same size where players can remove any number of tokens. That means that the first two levels of the game tree always look the same.



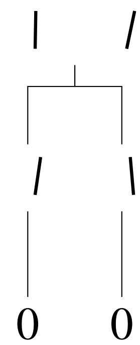
In NIM, we can consider each pile to be its own position, with multiple piles being a sum of positions. For example:

\ \\| = | + //|

or:

/ // \// = \ + \/ /|/ = | + \\ + \\/

We can find the outcome class of a game sum in the same way we found the outcome class of a single position: By using a game tree! Each of the options from the top includes just a move on one of the components.



What is the outcome class $o($ \ / $)$?

We would really like to find values to which different sets of positions are *equivalent*. By equivalent we mean that they always behave the same when we add them to other games. Positions G and H are equivalent if we can add the same game to both of them and both sums are always in the same outcome class. To use our notation:

$$\forall J \;:\; \text{if } o(G+J) = o(H+J), \text{then we say that } G \text{ is equivalent to } H, \text{ or } G = H$$

Math Diversion: Quantifiers
Mathematicians use symbols to abbreviate things that we tend to write often. Sometimes you'll see what look like upside-down or backwards letters. We've already seen the Greek letter $\in$ used for set inclusion. We can also use it backwards to show a set contains an element, like this:

$$S \ni x$$

Another couple of examples occur when we use *quantifiers* to talk about the existence of objects with particular properties. When we want to say that all elements in a certain set have a particular property then we use the *universal quantifier* ∀. For example

$$\forall x \in \mathbb{R}(x/2 \in \mathbb{R})$$

is read as "For any real number x, half of x is also a real number." Similarly, we use the *existential quantifier* ∃ to denote that an object with a particular property exists:

$$\exists n \in \mathbb{Z}(n = n^2)$$

says, "There exists an integer n that is equal to it own square." Note that both 0 and 1 qualify but the statement is true because there is *at least one* such integer.

We can combine quantifiers in long statements to make statements like

$$\forall r \in \mathbb{Q} \exists n \in \mathbb{Z}(rn \in \mathbb{Z})$$

What does this statement say, and is it true?

A: *For any rational number there is an integer such that their product is an integer. It's true because we can multiply by the denominator of r and the result is the numerator, which is an integer.*

What about the following statement?

$$\exists x \in \mathbb{R} \forall y \in \mathbb{R}(x + y > x)$$

A: *"There is a real number x such that adding any other real number results in a greater number." This is false, since y can be negative. For example, no matter what x is, if $y = -1$ then $x + y < x$. So the statement is false.*

For impartial games there is a neat trick to determine whether two games are equal.

In this text, we include 0 as a Natural number. Thus, $\mathbb{N} = \mathbb{N} \cup \{0\} = \mathbb{Z}_{\geq 0}$. This is a point of contention among mathematicians, so it's customary to settle on a convention and stick with it. As Combinatorial Game Theorists, $0 \in \mathbb{N}$ simplifies a lot of statements.

We can describe each impartial game as the set of its options. For impartial games, we will say that a position is equal to the set of its options. For example, a NIM position with three sticks, |||\ = {0, \ , \| }.

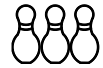
As mathematicians, we like to associate numbers to things whenever we can. It would certainly make things like addition and equivalence easier to handle. In [6] and [4], the authors realized that any two $\mathcal{P}$-Positions are equivalent, and hence we sometimes refer to numbers associated with impartial combinatorial games as *Grundy numbers* or *Grundy values*. Thus, whatever numbering system we use for games should assign the same number to all $\mathcal{P}$ Positions. Let's use 0. From here, we should probably assign the same number to every game that has precisely one move to a $\mathcal{P}$-position and no others. It makes sense to use 1 for that.

We can define these numbers recursively. Every leaf node is a $\mathcal{P}$-Position so assign it a 0. Any parents of only $\mathcal{N}$-Positions are also $\mathcal{P}$-Positions, so they get assigned 0, as well. Every parent of a 0 that has no other children gets a 1. In general, let's assign every node a number equal to the smallest non-negative integer assigned to its children. This is called the *minimum excludant*, abbreviated *mex*. So, for example, $mex\left(\{0,1\}\right) = 2$ and $mex\left(\{1,2\}\right) = 0$.

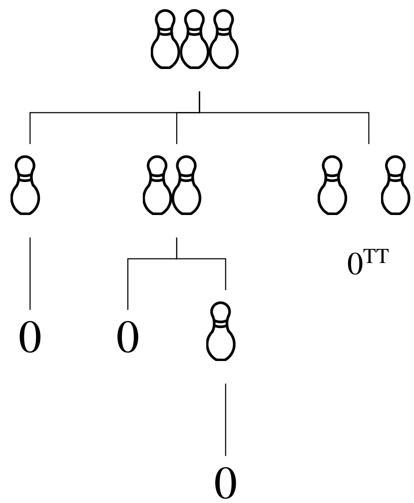
Unfortunately, by using positive integers, we run into conflict with values of partisan games we'll see in section 3.1. In order to differentiate, we will put a $*$ in front of the number. For example, $*5$ is the value of the impartial game with options: $*0, *1, *2, *3$, and $*4$. This means that $*0$ is the same as 0, so we'll just call it that. Additionally, $*1$ is so common that we will usually just refer to it as $*$.

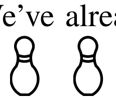
Look back at the trees we built for NIM. If there are no remaining sticks then we have a $\mathcal{P}$ Position and its Grundy number is 0. If there is a single stick then its only option is 0. Since $mex\left(\{0\}\right) = 1$, a single stick has a Grundy number of 1 and a value of $*$. What about a pile of ten sticks? It has options with any number of sticks less than ten. Since $mex\left(\{0,1,2,3,4,5,6,7,8,9\}\right) = 10$, it gets a Grundy number of ten (and a value of $*10$). In general, a pile of n-many sticks has Grundy number n and value $*n$. That makes things easy!

Because NIM has such a nice, simple way to determine Grundy numbers, we also call these *nimbers*. Every impartial game has a nimber and behaves effectively the same as a NIM pile. Let's formalize that to define *equality for impartial* *games*: we say that for two impartial games, G and H, $G = H$ if and only if both G and H have nimber k.[10] Furthermore, in that case, we say $G =* k = H$.

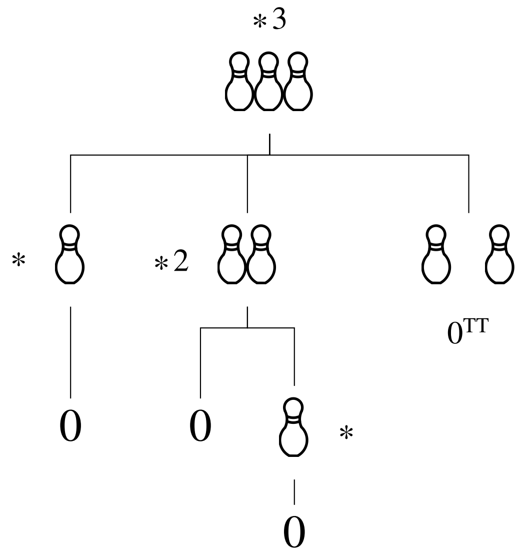
Let's use everything we learned to find the nimber value of  .

First, we draw out the game tree:



(We've already labelled the value for the Tweedledum-Tweedledee situation with  .) Next, we work our way up from the bottom, using the mex-rule to determine the nimbers.

[10]In Chapter 2, we'll learn that this definition doesn't work for partisan games. We'll present a more comprehensive on in Section 2.4.



For impartial games we will use notation similar to sets but with an asterisk on the left. That set will contain all the move options for that position. For example, this means that the root of that previous tree is equal to ${}^{*}\{\, *, *2, 0 \,\}$, which is equal to $*3$.

Computational Corner: Mex
Let's consider writing a function to calculate the mex of a list of integers.[a]
We want it to act like this:

```
>>> mex([1, 3, 5])
0
>>> mex([1, 0, 2])
3
```

We can write this with a simple loop that checks each non-negative integer in increasing order:

```
def mex(integers):
    '''Returns the mex (minimum excluded value) of integers.''
    num = 0 # next number to find
    while num in integers:
        num += 1
```

```
    return num
```

This runs in $O(n^2)$ time in the worst case. In Exercise 0.6.54 , we challenge you to write a mex function that runs in $O(n \log(n))$ time.

[a] We are using a list instead of a set as we expect more readers to be familiar with lists.

Consider two games, $G_1 = {}^*\{\,0, *, *2, *3\,\}$ and $G_2 = {}^*\{\,0, *, *2, *3, *5, *6, *7, *9\,\}$. Both of these have a nimber of $*4$, so, by our prior definition of equality (for impartial games), we say that both $G_1 =* 4$ and $G_2 =* 4$, (so $G_1 = G_2$.) What happens if we add G_1 and G_2?

Thankfully, $G_1 + G_2 = 0$. We can use a modified Tweedledum-Tweedledee strategy to prove this. If the next player chooses one of the moves to a nimber of $*3$ or less, then the previous player can proceed as in Tweedledum-Tweedledee by choosing the mirror move on the other component of the sum. On the other hand, if the next player moves to one of the bigger-nimber positions ($*5$, $*6$, $*7$, or $*9$) then the previous player can choose to move from that position to $*4$, an option they must all have. This same argument works for all nimbers, the proof of which is Exercise 0.6.56.

We can use everything we've learned to quickly find the values for positions in SUBTRACTION. For example, if we know we're interested in finding the nimber for $\underset{\{1,2\}}{5}$, we can start a table and add to it:

k	0	1	2	3	4	5
$\underset{\{1,2\}}{k}$						

We know that a pile of 0 has no moves, and a pile of 1 has only a move to 0, so our first entries are:

k	0	1	2	3	4	5
$\underset{\{1,2\}}{k}$	0	$*$				

For the next values, since our subtraction set is $\{1, 2\}$, we can use the mex-rule on the previous two elements to find that value. This means we can quickly write in the next value as we go along. The next two values are:

k	0	1	2	3	4	5
$\underset{\{1,2\}}{\mathbb{k}}$	0	$*$	$*2$	0	$*$	$*2$

What happens when we continue this table all the way up to $\underset{\{1,2\}}{\mathbb{10}}$? What pattern emerges?

k	0	1	2	3	4	5	6	7	8	9	10
$\underset{\{1,2\}}{\mathbb{k}}$	0	$*$	$*2$	0	$*$	$*2$	0	$*$	$*2$	0	$*$

The pattern seems to be: $\underset{\{1,2\}}{\mathbb{k}} = \begin{cases} 0, & k \equiv 0 \bmod 3 \\ *, & k \equiv 1 \bmod 3 \\ *2, & k \equiv 2 \bmod 3 \end{cases}$

Let's prove that this pattern continues for all pile sizes. Note that if we happen to know the value for a pile of size n, then it's easy to see what the value is for a pile of size $n + 1$. When we want to prove something about progressively larger sets, it's often useful to employ *mathematical induction*.

Math Diversion: Proof by induction

Often we want to prove some property for all positive integers, e.g. that $3^n > n^2$ or that $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$. One way to do this is with a technique called *mathematical induction*. We demonstrate that a very simple case of a claim is true, and then show that whenever the claim is true for one integer, it is also true for the next. It's easiest to see this through an example.

Consider the claim that $\boxed{\sum_{k=1}^{n} k = \frac{n(n+1)}{2}}$. We want to show that this claim is true for any positive integer n. We can plug in some values of n to check, but since there are infinitely many possible values for n we can never check them all. Instead, let's rewrite this claim as a function of n:

$$P(n) : \sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

In other words,
$P(n)$ is the claim that $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$,

$P(12)$ is the claim that $\sum_{k=1}^{12} k = \frac{12(12+1)}{2}$, and
$P(☺)$ is the claim that $\sum_{k=1}^{☺} k = \frac{☺(☺+1)}{2}$.

We begin by demonstrating that a *base case* is true. This is the smallest possible value for n, and is usually 0 or 1. The base case is often very easy to prove.

Base Case: $P(1)$. $\sum_{k=1}^{1} k = 1$, and $\frac{1(1+1)}{2} = 1$. So they are equal when $n = 1$ and the base case is proven. ✓

Next, we assume that the claim is true for n and use that to show that it is also true for $n + 1$.

Inductive Assumption: We assume $P(n)$ is true and show that $P(n + 1)$ is true. If $P(n)$ is true then $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$. Consider $\sum_{k=1}^{n+1} k$. This equals $\sum_{k=1}^{n} k + (n + 1)$. By our inductive assumption, this equals $\frac{n(n+1)}{2} + (n + 1)$ which, when simplified, equals $\frac{(n+1)(n+2)}{2}$. Therefore, $\sum_{k=1}^{n+1} k = \frac{(n+1)(n+2)}{2}$ and, hence, $P(n + 1)$ is true. ✓

Now, since we have shown both that $P(1)$ is true and $P(n) \Rightarrow P(n + 1)$, we have proven the claim. □

An inductive proof is like climbing a staircase. If I can prove to you that I can get up the first step, and that I can get from any one step to the next one, then I've shown that I can reach every step.

Try to use induction to prove the other claim above, that $3^n > n^2$ for any positive integer n.

A:. *We proceed by induction. Let $P(n)$ be the statement $3^n > n^2$.*
Base Case: *$P(1)$. $3^1 = 3$ and $1^2 = 1$, so the base case is true.* ✓
Inductive Assumption: *Assume $P(n)$ is true. Consider 3^{n+1}. Since $3^{n+1} = 3 \cdot 3^n$, and by inductive assumption $3^n > n^2$, we know that $3^{n+1} > 3 \cdot n^2$. Also, when $n > 1$, $2n^2 > 2n + 1$, and hence $3 \cdot n^2 = n^2 + 2n^2 > n^2 + 2n + 1$. The right side simplifies to $(n+1)^2$, and therefore $3^{n+1} > (n + 1)^2$, and $P(n + 1)$ is proven.* ✓
Therefore the claim is true. □

Now we are ready to proceed with our proof.

Claim 0.6.1. *The game values for* $\underset{\{1,2\}}{\mathbb{k}}$ *are determined by* $\underset{\{1,2\}}{\mathbb{k}} = \begin{cases} 0, & k \equiv 0 \ mod\ 3 \\ *, & k \equiv 1 \ mod\ 3 \\ *2, & k \equiv 2 \ mod\ 3 \end{cases}$

for all $k \in \mathbb{N}$.

Proof. We proceed by induction on k. Let $P(k)$ be the statement in the claim.

Base Case: The game $\underset{\{1,2\}}{\mathbb{0}}$ has value 0, so the base case is true. ✓

Inductive Assumption: Assume that $P(k)$ is true, and consider a pile of size $k+1$. The only options are to the games $\underset{\{1,2\}}{\mathbb{k}}$ and $\underset{\{1,2\}}{\mathbb{k} - \mathbb{1}}$. Thus, $\underset{\{1,2\}}{\mathbb{k} + \mathbb{1}}$ has the Grundy value associated with the mex of these two options. By our inductive assumption, these values depend on the value of $(k+1)$ mod 3.

If $(k+1)$ mod $3 = 0$ then k mod $3 = 2$ and $(k-1)$ mod $3 = 1$, so the options have the values $*2$ and $*$. Since $mex\left(\{1,2\}\right) = 0$, $\underset{\{1,2\}}{\mathbb{k} + \mathbb{1}} = 0$.

Similarly, if $(k+1)$ mod $3 = 1$ or 2, then $\underset{\{1,2\}}{\mathbb{k} + \mathbb{1}} = *$ or $*\,2$, respectively. Therefore, $P(k+1)$ is true, and the claim is proven.

□

Computational Corner: Recursion with Subtraction

We may want to have some code to calculate these nimbers for Subtraction-{ }like this:

```
>>> n = get_subtraction_nimber(0, [1,2])
>>> print(n)
0
>>> get_subtraction_nimber(1, [1,2])
1
```

The `get_subtraction_nimber` function will certainly be recursive. Let's give it a shot:

```
def get_subtraction_nimber(tokens, subtraction_list):
    '''Returns the nimber of a Subtraction position.'''
    option_nimbers = []
    for subtraction in subtraction_list:
```

```
        option_tokens = tokens - subtraction
        if option_tokens >= 0:
            option_nimber = get_subtraction_nimber(
                                    option_tokens, subtraction_list)
            option_nimbers.append(option_nimber)
    return mex(option_nimbers)
```

Computational Corner: Dynamic Programming

Our previous version of `get_subtraction_nimber` is pretty slow. The main reason for this is that we are recalculating values we've already computed. To speed things up to do the calculations by hand, we used a table to keep track of those values. Let's do the same to our function.

We will generalize it by adding another parameter: a dictionary[a] of values we've already calculated:

```
def get_subtraction_nimber(tokens, subtraction_list, lookup={}):
    '''Returns the nimber of a position in a game of Subtraction.'''
    if tokens in lookup:
        # we already calculated this value
        return lookup[tokens]
    option_nimbers = []
    for subtraction in subtraction_list:
        option_tokens = tokens - subtraction
        if option_tokens >= 0:
            option_nimber = get_subtraction_nimber(option_tokens,
                                    subtraction_list, lookup)
                option_nimbers.append(option_nimber)
    nimber = mex(option_nimbers)
    lookup[tokens] = nimber # update the table
    return nimber
```

Note: We are including a default parameter for the lookup parameter in case we have old code that still calls the function with only two parameters.

Including a table as a shortcut like this is known as *dynamic programming*. Implementing it in a recursive function can dramatically speed up code, most notably when you are making multiple recursive calls. To modify a recursive function to employ dynamic programming usually requires four changes:

- Modify the parameter list to include the table.
- Check whether you already have the result for the current value in the table so you can make the shortcut.
- Include the table as an argument when making recursive calls.
- Add the new value to the table before you return it.

If you forget any one of these, you will miss out on the improvement!

[a] More about Python dictionaries in Python 3 here: `https://docs.python.org/3/tutorial/datastructures.html#dictionaries`. This is Python's version of a Hash Map or Hash Table, where looking up a value theoretically takes constant ($O(1)$) time. There is an excellent discussion of Python's implementation in this Stack Overlow Q& A: `https://stackoverflow.com/questions/114830/is-a-python-dictionary-an-example-of-a-hash-table`.

Exercises for 0.6

0) Write the following sentence using mathematical quantifiers: "If there is a natural number between 3 and 4, then it is equal to its own inverse."

1) Write the following sentence using mathematical quantifiers: "Every natural number has a real square root."

2) Write the following sentence using mathematical quantifiers: "If twice an integer is greater than its square root, then that integer is smaller than 2."

3) Write the following sentence using mathematical quantifiers: "Every even number has an even cube."

4) Write the following sentence using mathematical quantifiers: "There is a rational number in lowest terms for which the numerator is less than a third as big as its denominator."

5) Write the following sentence using mathematical quantifiers: "For any pair of numbers, if their difference is greater than 1, then their sum is less than twice the larger of the two."

6) Show using induction that for all $n \in \mathbb{Z}^+$, $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$.

7) Show that there is no greatest prime number. (Hint: Show there are greater than n-many primes for all n.)

⋆ **8)** Conjecture a formula for $\sum_{j=1}^{n} 2^j$ and prove your claim using induction. (Answer 0.6.8 in Appendix)

9) Conjecture a formula for $\sum_{j=1}^{n} \frac{1}{j(j+1)}$ and prove your claim using induction.

⋆ **10)** Draw the full game tree for . (Answer 0.6.10 in Appendix)

11) Draw the full game tree for .

⋆ **12)** What is $mex\left(\{0,1,2,3,4\}\right)$? (Answer 0.6.12 in Appendix)

13) What is $mex\left(\{1,2,3,4\}\right)$?

⋆ **14)** What is $mex\left(\{0,1,2,4\}\right)$? (Answer 0.6.14 in Appendix)

15) If $S = \{0,2,4\}$, what is $mex\left(S\right)$?

16) If $S = \emptyset$, what is $mex\left(S\right)$?

17) If $S = \{10,11,12,13,14,15\}$, what is $mex\left(S\right)$?

⋆ **18)** If $S = \{0,1,2,3,4\}$, and $T = \{5,6,7,8,9\}$ what is $mex\left(S \cup T\right)$? (Answer 0.6.18 in Appendix)

19) If $S = \{0,1,2,3\}$, and $T = \{2,3,4,5\}$ what is $mex\left(S \cup T\right)$?

20) If $S = \{0, 1, 2, 5, 6, 7\}$, and $T = \{0, 1, 4, 5, 8, 9\}$ what is $mex\left(S \cup T\right)$?

⋆ **21)** If $S = \{0, 1, 2, 5, 6, 7\}$, and $T = \{0, 1, 4, 5, 8, 9\}$ what is $mex\left(S \cap T\right)$? (Answer 0.6.21 in Appendix)

22) If $S = \{0, 2, 4, 6, 8\}$, and $T = \{1, 3, 5, 7, 9\}$ what is $mex\left(S \cap T\right)$?

23) If $S = \{0, 2, 3, 4, 5\}$, and $T = \{0, 1, 3, 4, 5\}$ what is $mex\left(S \cap T\right)$?

⋆ **24)** If $S = \{0, 2, 4, 6, 8\}$, what is $mex\left(S^C\right)$? (Answer 0.6.24 in Appendix)

25) If $S = \{5, 6, 7, 8, 9\}$, what is $mex\left(S^C\right)$?

26) If $S = \{4, 5, 6, 7\}$, what is $mex\left(S^C\right)$?

27) If $S \subseteq \mathbb{N}$, what is $mex\left(S^C\right)$? Prove your answer. Hint: consider s to be the smallest element of S.

⋆ **28)** If $S = \{2k \mid k \in \mathbb{N}\}$, what is $mex\left(S\right)$? (Answer 0.6.28 in Appendix)

29) If $S = \{2k \mid k \in \mathbb{N}\}$, and $T = \{3k \mid k \in \mathbb{N}\}$ what is $mex\left(S \cup T\right)$?

30) If $S = \{2k \mid k \in \mathbb{N}\}$, and $T = \{3k + 1 \mid k \in \mathbb{N}\}$ what is $mex\left(S \cup T\right)$?

⋆ **31)** What is $mex\left(\mathbb{N} \setminus \{56\}\right)$? (Answer 0.6.31 in Appendix)

32) Let $S = \{5k \mid k \in \mathbb{N}\}$. What is $mex\left(\mathbb{N} \setminus S\right)$?

⋆ **33)** Let $S = \{5k - 1 \mid k \in \mathbb{N}\}$. What is $mex\left(\mathbb{N} \setminus S\right)$? (Answer 0.6.33 in Appendix)

34) Let $S = \{2k \mid k \in \mathbb{N}\}$ and $T = \{5k \mid k \in \mathbb{N}\}$. What is $mex\left(S \setminus T\right)$?

⋆ **35)** Let $S = \{2k \mid k \in \mathbb{N}\}$ and $T = \{5k \mid k \in \mathbb{N}\}$. What is $mex\left((S \setminus T)^C\right)$? (Answer 0.6.35 in Appendix)

36) Let $S = \{2k \mid k \in \mathbb{N}\}$, $T = \{3k \mid k \in \mathbb{N}\}$, and $R = \{5k \mid k \in \mathbb{N}\}$. What is $mex\left(S \cup (T \setminus R)^C\right)$?

⋆ **37)** Simplify ${}^*\{ *, *3, *5, *7, *9 \}$ to a single nimber value. (Answer 0.6.37 in Appendix)

38) Simplify ${}^*\{ 0, *, *2, *3, *5, *6, *8, *9 \}$ to a single nimber value.

⋆ **39)** Simplify ${}^*\{ 0, *, 0, 0, * \}$ to a single nimber value. (Answer 0.6.39 in Appendix)

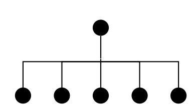
40) Simplify ${}^*\{ *, 0, *9, *8, *3, *8, *4 \}$ to a single nimber value.

⋆ **41)** $G = {}^*\left\{ 0, *2, *4, {}^*\{ 0, *2, *4 \} \right\}$ includes another impartial game's options written out. Simplify this to a single nimber value. (Answer 0.6.41 in Appendix)

42) $G = {}^*\left\{ {}^*\{ 0 \}, {}^*\{ * \}, *4, {}^*\{ 0, *2, *4 \} \right\}$. Find the nimber value of G.

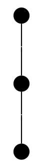
43) $G = {}^*\left\{ {}^*\{ 0, *, *2, *3, *4 \}, {}^*\left\{ *, *2, *3, {}^*\{ *, *6 \} \right\}, *4, {}^*\{ 0, *2, *4 \} \right\}$. Find the nimber value of G.

⋆ **44)** What is the value at the root of this impartial game tree?



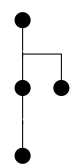
Justify your answer by labeling each node of the tree with its value. (This is a follow-up to exercise 0.2.0.) (Answer 0.6.44 in Appendix)

45)



Justify your answer by labeling each node of the tree with its value. (This is a follow-up to exercise 0.2.1.)

46) What is the value at the root of this impartial game tree? Justify your answer by labeling the nodes in the tree with the nimbers.



⋆ **47)** Let G be the position in SUBTRACTION-$\{1,2,3\}$ with a pile of 3. What is the nimber of G? Justify your answer by drawing the game tree and labeling each node with its nimber. (This is a continuation of exercise 0.2.5.) (Answer 0.6.47 in Appendix)

48) Let G be the position in SUBTRACTION-$\{1,2,3\}$ with a pile of 4. What is the nimber of G? Justify your answer by drawing the game tree and labeling each node with its nimber value. (This is a continuation of exercise 0.2.6.)

⋆ **49)** Using the table for $\underset{\{1,2\}}{\mathbb{k}}$ as a model, create a similar table to find the nimber of $\underset{\{1,2,3\}}{\mathbb{5}}$. (Answer 0.6.49 in Appendix)

50) Continue the table in exercise 0.6.49 to go all the way up to $\underset{\{1,2,3\}}{\mathbb{10}}$. What pattern do you see? When does each nimber seem to occur?

51) Prove, by strong induction, that the pattern you found in exercise 0.6.50 continues for all pile sizes.

⋆ **52)** Do the same as exercises 0.6.50 and 0.6.51 , except with the subtraction set $\{1, 3\}$ instead. (Answer 0.6.52 in Appendix)

53) Do the same as exercises 0.6.50 and 0.6.51 , except with the subtraction set $\{1, 4\}$ instead.

⋆ **54)** The version of the `mex` function given in the chapter is elegant partly because it is so short. Unfortunately, $O(n^2)$ is pretty inefficient. We can drastically speed it up to $O(n \log(n))$ by writing a bit more code. Rewrite `mex` to run in $O(n \log(n))$ time. (Hint: Python's built-in `sort` method for lists takes $O(n \log(n))$ time.) (Answer 0.6.54 in Appendix)

55) In the Dynamic Programming Computational Corner, we included four changes that have to be made to implement dynamic programming in a recursive function. For each of those changes, explain what will happen if a programmer forgets to implement just that part but does all the others.

⋆ **56)** In the text, we showed two different games, G_1 and G_2 that were both equal to $*4$. We showed that $G_1 + G_2 \in \mathcal{P}$, even though they weren't identical positions. Show that this works for any two equal (but not necessarily identical) positions. Prove that if $G = H$, then $G + H \in \mathcal{P}$. (Answer 0.6.56 in Appendix)

57) Prove the inverse of Exercise 0.6.56: for two impartial games G and H, if $G + H \in \mathcal{P}$ then $G = H$.

0.7. Nim Sums

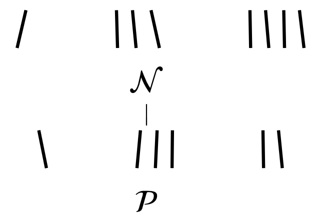
One for you and one for me
But one and one and one hardly three

- Nancy Sinatra[11], "Sugar Me"

Nimbers are for more than just determining whether two impartial positions sum to zero. Consider the following Nim position:

You can use the tactics we've learned so far to analyze the game tree and see that this is a $\mathcal{P}$-position. Alternatively, as we will learn next, we can simply use the three nimbers and see that $*1 + *2 + *3 = 0$.

Before we get too excited, let's notice that this math doesn't exactly work as we might want. Indeed, $*1 + *3 + *4 \neq 0$, because it has a zero option.



Since it's in $\mathcal{N}$, it must have a non-zero nimber value. Somewhat unexpectedly, $* + *3 + *4 = *6$. As we know, adding any two games with the same nimber value results in 0, meaning that \ /\\ |/|| /||\\|/ $\in \mathcal{P}$.

Thankfully, there is a trick to determine sums of nimbers ("nim sums") without trying out a bunch of game trees. Interestingly, this process uses binary representations and the logical XOR operator. We explain how to perform each part of this calculation.

In order to find the sum of two nimbers, we first have to represent the nimbers in binary: strings of only 1s and 0s. For any natural number, x, we can represent x as a sum of powers of two where we include each power either once or not at all.

Before we see an example, note that you're used to doing the same thing with powers of ten. The number $3,279$ is $3 \cdot 10^3 + 2 \cdot 10^2 + 7 \cdot 10^1 + 9 \cdot 10^0$. We are just so used to writing numbers in base ten that we do it without thinking about it. *Binary*, or *base two*, is the same but with powers of two.

As examples,

$$\begin{aligned} 5 &= 4 + 1 \\ &= 2^2 + 2^0 \\ &= 1 \cdot 2^2 + 0 \cdot 2^1 + 1 \cdot 2^0 \end{aligned}$$

$$\begin{aligned} 100 &= 64 + 32 + 4 \\ &= 2^6 + 2^5 + 2^2 \\ &= 1 \cdot 2^6 + 1 \cdot 2^5 + 0 \cdot 2^4 + 0 \cdot 2^3 + 1 \cdot 2^2 + 0 \cdot 2^1 + 0 \cdot 2^0 \end{aligned}$$

$$\begin{aligned} 91 &= 64 + 16 + 8 + 2 + 1 \\ &= 2^6 + 2^4 + 2^3 + 2^1 + 2^0 \\ &= 1 \cdot 2^6 + 0 \cdot 2^5 + 1 \cdot 2^4 + 1 \cdot 2^3 + 0 \cdot 2^2 + 1 \cdot 2^1 + 1 \cdot 2^0 \end{aligned}$$

It may be helpful to commit some powers of 2 to memory[12]:

2^0	2^1	2^2	2^3	2^4	2^5	2^6	2^7	2^8	2^9	2^{10}
1	2	4	8	16	32	64	128	256	512	1,024

After generating these sums of powers of two, the important part is to know where the 1s and 0s are. Then we can write these out in binary, including a subscript of 2 to indicate that the number should be considered a binary number.

Back to our three examples:

- $5 = 1 \cdot 2^2 + 0 \cdot 2^1 + 1 \cdot 2^0$, so $101_2 = 5$.
- $100 = 1 \cdot 2^6 + 1 \cdot 2^5 + 0 \cdot 2^4 + 0 \cdot 2^3 + 1 \cdot 2^2 + 0 \cdot 2^1 + 0 \cdot 2^0$, so $1100100_2 = 100$.
- $91 = 1 \cdot 2^6 + 0 \cdot 2^5 + 1 \cdot 2^4 + 1 \cdot 2^3 + 0 \cdot 2^2 + 1 \cdot 2^1 + 1 \cdot 2^0$, so $1011011_2 = 91$.

[12] Computer science students should memorize their powers of 2 up through 2^{16}.

Similar to how we normally refer to each place value as the “ones”, “tens”, “hundreds” (going from right-to-left), we refer to the different binary places as “ones”, “twos”, “fours”, “eights”, etc. As you might expect, each natural number can be written in binary in exactly one way.

Once we have the binary representations of the two nimbers, we can use a bit-wise XOR ($\oplus$) to add them together. “Bit-wise” means we will calculate the sum digit-by-digit. To perform the XOR, we can use this table:

x	0	0	1	1
y	0	1	0	1
$x \oplus y$	0	1	1	0

There are alternative ways to think of this, which even work for adding more than two bits:

- The resulting bit is a 1 if and only if there are an odd number of 1s being added.
- XOR is just addition without carrying.

Use whichever works best for you.

As an example, let’s add $*6$ and $*5$. First, we convert 6 and 5 to binary. $6 = 110_2$ and $5 = 101_2$. Then we perform the XOR:

$$\begin{array}{rr} & 110 \\ \oplus & 101 \\ \hline & 011 \end{array}$$

The final step is to convert the 011_2 back to decimal. As before, each digit corresponds with a power of 2, so we just add those together.

$$\begin{aligned} 011_2 &= 0 \cdot 2^2 + 1 \cdot 2^1 + 1 \cdot 2^0 \\ &= 2^1 + 2^0 \\ &= 2 + 1 \\ &= 3 \end{aligned}$$

Thus, $*6 + *5 = *3$.

We can use this to quickly find the nimber of any NIM game. For example to find the value of, \   //   /\| |/|| \\//\ :

$$\begin{array}{rcr} *1 & \longrightarrow & 001_2 \\ *2 & \longrightarrow & 010_2 \\ *3 & \longrightarrow & 011_2 \\ *4 & \longrightarrow & 100_2 \\ + *5 & \longrightarrow & \oplus\, 101_2 \\ \hline *1 & \longleftarrow & 001_2 \end{array}$$

With this shortcut, when evaluating the sum of multiple impartial game positions, it is often more difficult to determine the individual nimbers than it is to add those nimbers together.

This brings us to what might seem like an obvious result but is still worth proving rigorously here.

Claim 0.7.1. $*k = *l \Leftrightarrow k = l$

Proof. Firstly, it is clear that if $k = l$ then $*k = *l$. Now, assume that $*k = *l$. Adding $*k$ to both sides yields $*k + *k = *k + *l$. Since the left side is 0, this means that the right side, $*k + *l$, must also be 0. By the definition of the nim sum, this is equal to 0 if and only if the binary expansions of k and l are identical, which is only true when $k = l$. □

Computational Corner: Adding Nimbers
How long does it take to add two nimbers? To perform the calculation by hand, we have to convert the number to binary, XOR each digit, then convert it back to decimal. Computers already keep numbers stored in binary, so they get to skip that step. The XOR process, then, is constant for each digit, so we only need to know how many digits are in the number. This is exactly $\lceil \log_2(x) \rceil$. Thus, summing two numbers, x_1 and x_2, takes $O(\log(x_1) + \log(x_2))$ time.

Don't go writing your own function, however! Most programming languages have a built-in bitwise-XOR operator. In Python, ^ will do the trick:

```
>>> 3 ^ 6
```

5

Exercises for 0.7

★ **0)** What is the nimber value $*3+*2$? (Answer 0.7.0 in Appendix)

1) What is the nimber value $*5+*3$?

★ **2)** What is the nimber value $*4+*2$? (Answer 0.7.2 in Appendix)

3) What is the nimber value $*6+*11$?

★ **4)** What is the nimber value $*5+*11$? (Answer 0.7.4 in Appendix)

5) What is the nimber value $*20+*21$?

★ **6)** What is the nimber value $*20+*31$? (Answer 0.7.6 in Appendix)

7) What is the nimber value $*20+*41$?

8) What is the nimber value $*20+*37$?

★ **9)** What is the value of $* + *5 + *6$? (Answer 0.7.9 in Appendix)

10) What is the value of $*3+*4+*5$?

★ **11)** What is the value of $*10+*20+*25$? (Answer 0.7.11 in Appendix)

12) What is the value of $*10+*20+*30+*40$?

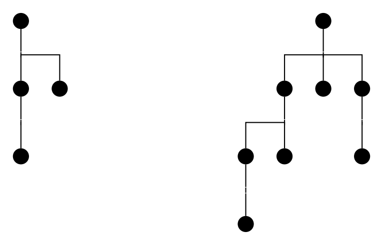
★ **13)** What is the value of $\sum_{i=1}^{7} *i = * + *2+*3+*4+*5+*6+*7$? (Answer 0.7.13 in Appendix)

14) What is the value of $\sum_{i=1}^{9} *i = * + *2+*3+*4+*5+*6+*7+*8+*9$?

⋆ **15)** Using what you learned in exercises 0.7.13 and 0.7.14 , what is the value of $\sum_{i=1}^{4k+1} *i = * + *2 + *3 + \cdots + *(4k) + *(4k+1)$? (Answer 0.7.15 in Appendix)

16) Using what you learned in exercises 0.7.13 , 0.7.14 , and 0.7.15 , what is the value of $\sum_{i=1}^{4k+3} *i = * + *2 + *3 + \cdots + *(4k+2) + *(4k+3)$?

17) Find the (nimber) value of the game that's the sum of these two trees.



⋆ **18)** Find an option of the game

/ / \ | / \ / / \ \   \ \ / | / | / \ |

i.e. $(3,7,9)$, that is in $\mathcal{P}$. (Answer 0.7.18 in Appendix)

19) Find an option of the game

\ | \ | | | | / / \ \ / | \ \ \ \ | | | / \ \ \ /

i.e. $(4,9,12)$, that is in $\mathcal{P}$.

⋆ **20)** Find an option of the game

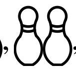
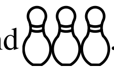
| \ / | | \ | /   \ / \ / |

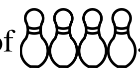
i.e. $(6,2,5)$, that is in $\mathcal{P}$. (Answer 0.7.20 in Appendix)

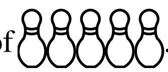
21) Find an option of the game

i.e. $(13,7,9)$, that is in $\mathcal{P}$.

22) Explain how to always find an option of any NIM position in $\mathcal{N}$ that is in $\mathcal{P}$.

⋆ **23)** Find the nimber values of , , and . (Answer 0.7.23 in Appendix)

⋆ **24)** Find the nimber value of . You will likely need to use your answer to exercise 0.7.23 . (Answer 0.7.24 in Appendix)

25) Find the nimber value of . You will likely need to use your answer to exercises 0.7.23 and 0.7.24 .

26) Use your answers to exercises 0.7.23 through 0.7.25 and find more KAYLES nimber values to fill in this table of KAYLES nimbers:

# pins	0	1	2	3	4	5	6	7	8	9	10
value											

27) Using the `mex` function we've previously written, write a function `print_kayles_nimbers` that takes a single non-negative integer, `n`, and prints the nimber values of unbroken rows of KAYLES pins from zero to `n`. E.g., `print_kayles_nimbers(3)` should print:

```
Kayles with 0 pins equals *0
Kayles with 1 pins equals *1
Kayles with 2 pins equals *2
Kayles with 3 pins equals *3
```

(Hint: ^ is the built-in operator for bitwise xor. Double hint: Our solution uses dynamic programming and a separate function `get_kayles_nimber`.)

28) Prove that for any KAYLES position, G, with n unbroken pins: $n = 0 \Leftrightarrow G = 0$.

0.8. Other Impartial Games

In order to play this next game, we will need to know some things about booleans and boolean arithmetic.

Unlike normal arithmetic with numbers, *booleans* only include two values: true and false, which we will denote as T and F, respectively. Just as with numbers, these two values have real-world meaning which we expect you are familiar with.

We can have boolean variables, just as we have numeric variables, e.g. x_1 and x_2 may be variables that could each be either T or F. Then, instead of addition and multiplication, two of our main operators are: *or* ($\vee$) and *and* ($\wedge$). Thus the expression: $x_1 \vee x_2$ means "x_1 or x_2", and we can determine the truth value of that expression based on the possible cases of the two values. This is described in something called a *truth table*:

x_1	x_2	$x_1 \vee x_2$
F	F	F
F	T	T
T	F	T
T	T	T

The left columns are arranged to provide for every possible combination of T and F values of the variables. Then the column on the right provides the value of the expression above for those values. For example, the third row says that when x_1 is T and x_2 is F, the value of $x_1 \vee x_2$ is T.

Similarly, here is the truth table for And:

x_1	x_2	$x_1 \wedge x_2$
F	F	F
F	T	F
T	F	F
T	T	T

If we have a more-complicated formula, it is common to use extra columns to compute the partial result before the whole thing. For example, to calculate $(x_1 \vee x_2) \vee (x_1 \wedge x_2)$, we might use:

x_1	x_2	$x_1 \vee x_2$	$x_1 \wedge x_2$	$(x_1 \vee x_2) \vee (x_1 \wedge x_2)$
F	F	F	F	F
F	T	T	F	T
T	F	T	F	T
T	T	T	T	T

There is also the negation operator, which is used to flip from true-to-false and vice-versa. In this book, we use a line over the variable or expression being negated, e.g. $\overline{x_1 \vee x_2}$. In other places, you may see it written with a $\sim$ or $\neg$ before an expression, e.g. $\sim (x_1 \vee x_2)$ or $\neg(x_1 \vee x_2)$. The negation operator gives us (nearly) the most simple truth table:

x_1	$\overline{x_1}$
F	T
T	F

Then, for example, we could create the truth table for $\overline{x_1 \vee x_2}$:

x_1	x_2	$x_1 \vee x_2$	$\overline{x_1 \vee x_2}$
F	F	F	T
F	T	T	F
T	F	T	F
T	T	T	F

There are a variety of combinatorial game rulesets that use boolean formulas. In these games, players are usually changing or setting boolean variables on each of their turns. It is also common for these games to be in Conjunctive Normal Form (CNF) meaning:

- Only literals (instances of variables) may be negated, not larger expressions.

- The entire formula consists of a bunch of clauses with the and-operator between them. Each clause is in parentheses, so the whole formula has a structure like: $(\cdots) \wedge (\cdots) \wedge \cdots \wedge (\cdots)$

- Each clause consists of literals or negated literals with the or-operator between them, e.g. $(x_1 \vee \overline{x_2} \vee \overline{x_3})$

Here is an example of a boolean CNF-formula with three clauses and four variables:

$$(x_1 \vee \overline{x_2} \vee x_3) \wedge (\overline{x_1} \vee \overline{x_3} \vee x_4) \wedge (x_2 \vee x_3 \vee \overline{x_4})$$

AVOID TRUE

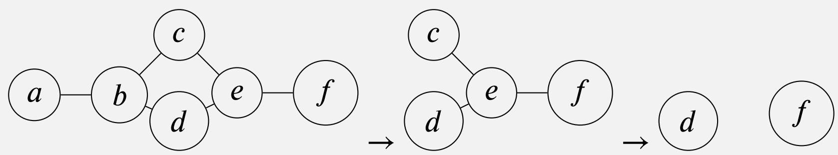
AVOID TRUE is a game played with a list of boolean variables $(x_1, x_2, \ldots, x_n)$ and a CNF, f, using those variables that has no negations. All variables begin the game set to False. A turn consists of picking one variable that is still False and flipping it to True, such that the whole formula still evaluates to False. (A variable cannot be chosen if flipping that would cause the formula to become True.)
To simplify positions, we will remove clauses that are already satisfied and list extra variables afterwards.

$$(x_1 \vee x_2) \wedge (x_2 \vee x_3 \vee x_4) = F$$
$$(x_1 \vee x_2) \wedge (x_2 \vee T \vee x_4) = F$$
$$(x_1 \vee x_2) \wedge (x_2 \vee T \vee T) = F$$

The first move is to flip x_3, making the whole second clause true. The second flips x_4, which leaves no further moves.

Node Kayles

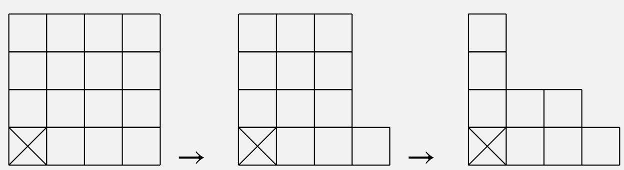
Node Kayles is a game played on an undirected graph. On the current player's turn, they choose a vertex. Then, the graph is altered by removing that vertex and all adjacent vertices. When there are no more vertices, then there are no more moves.



The first move chooses a, the second chooses c.

Chomp!

Chomp (sometimes stylized *Chomp!*) is an impartial game played on a grid with squares labeled with integer coordinates from $[1, n] \times [1, m]$. On their turn a player chooses a remaining square with label (x, y) and removes it along with all remaining squares of the form (x_1, y_1) such that $x_1 \geq x$ and $y_1 \geq y$. The player who removes square $(1, 1)$ loses.



The first play is at $(4, 2)$, removing $(4, 2), (4, 3)$, and $(4, 4)$. The next play is at $(2, 3)$, removing that position along with all pieces above and to the right of this piece.

Chomp is a simple game to play but difficult to strategize for as the board size increases. In fact, no strategy is known for Chomp in general. However, we *do* know who can always win on a rectangular board!

Claim 0.8.1. *Any rectangular board* $[1, n] \times [1, m]$ *in* CHOMP *with more than a single square is in* $\mathcal{N}$.

Proof. We proceed by contradiction. Assume that the claim is false, and that there is a strategy S whereby the second player can win the game. So, by assumption, for any move (x, y) by the first player there is a strategic response (r, s) that the second player can make on their way to win the game. In particular, if the first player plays on square (m, n), removing only that square and nothing else, then the second player's response of square (r_1, s_1) will lead to a win for them following S. However, since $r_1 \leq n, s_1 \leq m$, there is nothing preventing the first player from making this their first move instead, and then following the same strategy S that the second player had planned. Note that the result of removing (r_1, s_1) on the first move is identical to the result of removing first (m, n) and then (r_1, s_1). Thus, if S is a strategy for the second player to win, then it is also a strategy for the first player to win. Hence there is no such strategy, and the game must be in $\mathcal{N}$. □

You may be thinking to yourself that this method is strange, and it is! We have proved that the game is in $\mathcal{N}$ using a *strategy stealing* argument. What's more, we have used a *non-constructive* proof.

Math Diversion: Constructive and non-constructive proof
We often prove the existence of something by finding it and presenting it to our audience. For example, we can prove that there is a multiplicative inverse for every $x \in \mathbb{R}_{\neq 0}$ by simply pointing out that $1/x \in \mathbb{R}_{\neq 0}$ and noting that $x \cdot 1/x = 1$. Or, as above, we explain that every positive integer has a Zeckendorf representation by demonstrating how to find it. This is called a *constructive proof* because we have demonstrated that something exists by finding it.

But we can also prove claims using *non-constructive* methods. For example, consider the following claim about prime numbers.

Claim 0.8.2. *There are infinitely many prime numbers.*

Proof. We proceed by contradiction. Assume that there are only finitely many prime numbers, and let p be the largest. Let

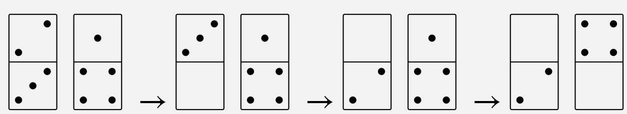
$$n = p! = p(p-1)(p-2)\cdots 1 + 1.$$

Note that no integer m greater than 1 and less than or equal to p divides n without remainder, (which we write as $m|n$). But by the Fundamental Theorem of Arithmetic, every positive integer has a prime factorization. So there much be *some* prime number q such that $q|n$, even if $n = q$ is itself prime. But we've seen that $q \notin [2, p]$, and so $q > p$. Thus, we have discovered the existence of a prime number greater than p, which we took as the largest prime. Therefore, our assumption is false, and there are infinitely many primes □

Notice that in our proof we did not actually list infinitely many primes. Nor did we even find a prime larger than p. We simply demonstrated that such a prime exists. This is a non-constructive proof that there are infinitely many primes.

DOMINIM

DOMINIM is a NIM variant in which players are presented with a collection of dominoes. Each domino has a number of pips on top and a possibly different number on the bottom. Players make NIM moves on the set of domino tops, and once a domino is played it is flipped, so the bottom becomes the new top. The game ends when all tops contain zero pips.



The first player removes both pips from the lefthand domino then flips it. The second player responds by removing one from the same domino and flips it again, making in unplayable. The next player is forced to remove the only pip in the righthand domino, leaving the following player a single heap of four to remove and win.

Exercises for 0.8

⋆ **0)** Complete the following truth table for $x_1 \wedge (x_2 \vee x_3)$.

x_1	x_2	x_3	$x_2 \vee x_3$	$x_1 \wedge (x_2 \vee x_3)$
F	F	F	F	F
F	F	T		
F	T	F		
F	T	T		
T	F	F		
T	F	T		
T	T	F		
T	T	T		

(Answer 0.8.0 in Appendix)

1) Following the pattern of rows for exercise 0.8.0 , create a truth table for $(x_1 \wedge \overline{x_2}) \vee (x_2 \wedge \overline{x_3})$. You may use as many intermediate columns as you like.

⋆ **2)** How many rows would we need in a truth table for a formula with four boolean variables? (Answer 0.8.2 in Appendix)

3) How many rows do we need in a truth table for a formula with n boolean variables?

⋆ **4)** Consider AVOID TRUE played on the formula $(x_1 \vee x_2) \wedge x_3$. Assume the players are planning to play on variables in this order: x_1, x_2, then x_3. Evaluate the formula after each (attempted) move (and show your work). What is the first of these moves that can't be made because it will be illegal? (Answer 0.8.4 in Appendix)

5) Consider AVOID TRUE played on the formula $(x_1 \vee x_2) \wedge x_3$. Assume the players are planning to play on variables in this order: x_3, x_1, then x_2. Evaluate the formula after each (attempted) move (and show your work). What is the first of these moves that can't be made because it will be illegal? (This is just like exercise 0.8.4, but with the moves done in a different order.)

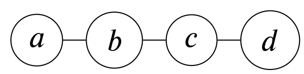
6) Consider AVOID TRUE played on the formula $(x_1 \vee x_2) \wedge x_3$. Is the game in $\mathcal{N}$? If so, what is the winning move? (Recommended: complete exercises 0.8.4 and 0.8.5 first.)

⋆ **7)** Consider AVOID TRUE played on the formula $(x_1 \vee x_2) \wedge (x_2 \vee x_3 \vee x_4)$. Which variables can the first player *not* choose? (Answer 0.8.7 in Appendix)

⋆ **8)** Consider AVOID TRUE played on the formula $(x_1 \vee x_2) \wedge (x_2 \vee x_3 \vee x_4)$. What is the outcome class of this position? (This is a continuation of exercise 0.8.7 .) (Answer 0.8.8 in Appendix)

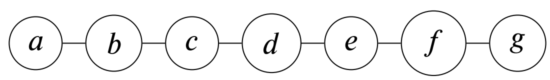
9) Consider AVOID TRUE played on the formula $(x_1 \vee x_2) \wedge (x_2 \vee x_3 \vee x_4)$. What is the nimber value of this position? (This is a continuation of exercise 0.8.8 .)

⋆ **10)** Use a game tree to determine the outcome class of NODE KAYLES on a path with four vertices.



Hint: combine moves that result in equivalent graphs into one option instead of separate. (Answer 0.8.10 in Appendix)

11) In many positions, a player can find a winning move without analyzing the whole game tree. Recalling the Tweedledee-Tweedledum strategy, what is the winning move for the first player on a NODE KAYLES path with seven vertices?



Justify your answer.

⋆ **12)** Find the outcome class and winning strategy for CHOMP on any rectangular board of size $2 \times n, n \geq 2$. (Answer 0.8.12 in Appendix)

13) Determine the game value of a single row CHOMP position of length n.

⋆ **14)** Determine the game value of the CHOMP positions $2 \times 2, 2 \times 3$, and 2×4. (Answer 0.8.14 in Appendix)

⋆ **15)** Play $2 \times n$ CHOMP with someone else for a few different values of n and win every time. (Answer 0.8.15 in Appendix)

⋆ **16)** Find all single-domino DOMINIM positions in $\mathcal{P}$. (Answer 0.8.16 in Appendix)

17) Prove that the Grundy value of a DOMINIM position with t on top and b on the bottom is equal to $b + 1$

0.9. Sequences of impartial game values

Patterns show up all over Discrete Mathematics. We've seen then in $\mathcal{P}$-positions of SUBTRACTION and in the game values of other impartial games. In this chapter we will look at just a few ways of identifying and characterizing sequences.

FIBONACCI NIM

FIBONACCI NIM is played identically to NIM with two additional restrictions: The first player may not remove all the sticks, and no player may remove more than twice the number of sticks removed on the previous turn.

\ / | | \ \ | | → | \ | \ \ / → | \ \ → 0

The first player removes 2 sticks. Since the second player can remove at most $2 \cdot 2 = 4$, they do so. The first player can remove the remaining $3 < 2 \cdot 4$.

There are myriad modified versions of NIM. Why is FIBONACCI NIM so interesting, and where does it get its name? First we need a short mathematical diversion.

Math Diversion: Zeckendorf representations
Recall that the *Fibonacci Numbers* are the values in the sequence recursively

defined by

$$F(n) = \begin{cases} 1 & n = 0 \text{ or } 1 \\ F(n-1) + F(n-2) & \text{otherwise} \end{cases}$$

The first few are $1, 1, 2, 3, 5, 8, 13, 21, 34 \ldots$. Just as every natural number can be written as a unique sum of distinct powers of two (its binary representation), and, as long as it's greater than one, as a unique product of primes (its prime decomposition), it can also be written as a unique sum of non-consecutive Fibonacci numbers. For example, $20 = 13 + 5 + 2$. This is known as a number's *Zeckendorf representation*, after Édouard Zeckendorf [**?**]. What are the Zeckendorf representations for 30, 40, and 50?

A: $30 = 21 + 8 + 1, 40 = 34 + 5 + 1$, *and* $50 = 34 + 13 + 3$

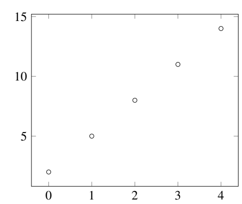
Imagine that you have written a number n as the sum of Fibonacci numbers, but some are consecutive. Think a bit on your own about how you could turn your sum into a Zeckendorf representation for n.

A: *Note that if your sum contains both $F(k)$ and $F(k+1)$ then you can replace the pair of them with $F(k+2)$. This process can be repeated until no consecutive Fibonacci numbers remain.*

Now let's return to FIBONACCI NIM. Before determining all of the $\mathcal{P}$-positions let's find a winning strategy directly. Say that you have a heap of n sticks. If this game is in $\mathcal{N}$ then there is a move that will take it to $\mathcal{P}$. If $n = F(i_1) + F(i_2) + \cdots + F(i_k)$ is the Zeckendorf representation for n, then remove $F(i_1)$ sticks from the heap if possible. What remains is $n - F(i_1)$ sticks, which has its own Zeckendorf representation of $F(i_2) + \cdots + F(i_k)$. Note that since i_1 and i_2 are not consecutive integers, and $F(i_1), F(i_2)$ are not consecutive Fibonacci numbers, the value of $F(i_2)$ is more than twice as large as $F(i_1)$, (remember that $F(k) + F(k+1) = F(k+2)$, and $F(k+1) > F(k)$, so $F(k+2) > 2 \cdot F(k)$). This means that if you are able to remove $F(i_1)$ sticks then your opponent cannot legally remove $F(i_2)$ sticks. It is possible to show that no matter what move your opponent makes, what remains has a Zeckendorf representation with a smallest value that you will be able to remove on your subsequent turn. It turns out that the only $\mathcal{P}$-positions are heaps exactly the size of a Fibonacci number.

The Fibonacci numbers aren't the only sequence worth further examination. So far, we have seen sequences pop up in a number of places, both in the distribution of $\mathcal{P}$positions and grundy values of games. Let's take some time now to look at various kinds of sequences. The more comfortable we become with them, the better equipped we will be to predict values and strategies of other games.

You are already familiar with *arithmetic sequences*. These are sequences where each term differs from the previous term by a constant. For example, consider the sequence $2, 5, 8, 11, 14, \ldots$. Plotting these terms shows a straight line.



If we examine the differences between successive terms we see a constant.

$$2 \underbrace{\qquad}_{+3} 5 \underbrace{\qquad}_{+3} 8 \underbrace{\qquad}_{+3} 11 \underbrace{\qquad}_{+3} 14 \ldots$$

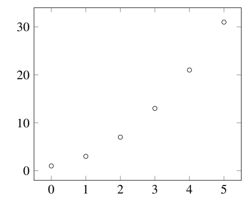
This sequence is *arithmetic* because we add a constant to each term to reach the next. Since each term is 3 greater than the previous, we can write the sequence using our recurrence relation notation as

$$a_n = \begin{cases} 2 & \text{if n=0} \\ a_{n-1} + 3 & \text{otherwise} \end{cases}$$

Of course, we can also write it in a *closed form*, which does not require, for example, determining a_{99} in order to determine a_{100}:

$$a_n = 2 + 3n$$

What if we encounter the sequence $\{b_n\} = 1, 3, 7, 13, 21, 31, \ldots$? Another plot shows us that we need to think about something non-linear.



Can we find a closed form in a similar way? Let's examine the differences again:

$$1 \underbrace{\qquad}_{+2} 3 \underbrace{\qquad}_{+4} 7 \underbrace{\qquad}_{+6} 13 \underbrace{\qquad}_{+8} 21 \underbrace{\qquad}_{+10} 31 \dots$$

The differences are no longer constant, so the sequence is not arithmetic. However, the differences of the differences *is* constant:

$$2 \underbrace{\qquad}_{+2} 4 \underbrace{\qquad}_{+2} 6 \underbrace{\qquad}_{+2} 8 \underbrace{\qquad}_{+2} 10 \dots$$

We say that the original sequence $1, 3, 7, 13, 21, 31, \dots$ is Δ^2-constant, (an arithmetic sequence is Δ^1-constant). What this tells us is that a closed formula for the sequence must have an n^2 term. So,

$$b_n = An^2 + Bn + C.$$

We need only solve for A, B, and C, so let's plug in the first few terms and see what happens.

$$\begin{aligned} b_0 &= 1 = A0^2 + B0 + C = C \\ b_1 &= 3 = A1^2 + B1 + C = A + B + C \\ b_2 &= 7 = A2^2 + B2 + C = 4A + 2B + C \end{aligned}$$

We have three equations and three unknowns, which we can solve using elimination or substitution. We will work through this case, then leave future examples to the reader. Note that $C = 1$ so we substitute it in immediately.

$$\begin{array}{lll} 3 = A + B + 1 & 2 = A + B \quad (\times 2) & 4 = 2A + 2B \\ 7 = 4A + 2B + 1 & 6 = 4A + 2B & (-)6 = 4A + 2B \\ \hline & & -2 = -2A \end{array}$$

So $C = 1$ and $A = 1$. Substituting both of these into either the second or third equation yields $B = 1$. Hence, $b_n = n^2 + n + 1$. This method of finding a closed formula for a sequence is called *polynomial fitting*, and it works on any sequence that isΔ^k-constant for some $k \in \mathbb{N}$.

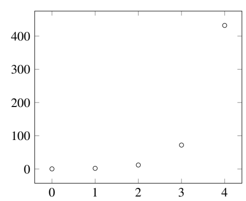
It's worth recognizing a very important quadratic sequence called the *triangular numbers*:

$$\{T_n\} = 0, 1, 3, 6, 10, 15, 21, 28, 36, \ldots$$

These show up in a lot of mathematical places, especially in combinatorial situations. You should solve this yourself using polynomial fitting to see that $T_n = \frac{(n)(n+1)}{2}$. You may have seen them in Calculus as the sum of the natural numbers up to n, in Graph Theory as the number of edges in a graph on $(n + 1)$ nodes, or just the number of objects needed to be arranged into a triangular shape.

What if our sequence isn't Δ^k constant for *any* $k \in \mathbb{N}$? Consider the sequence, its differences, and the plot below.

$$\{c_n\} = \frac{1}{3}\underbrace{\quad}_{+\frac{5}{3}}2\underbrace{\quad}_{+10}12\underbrace{\quad}_{+60}72\underbrace{\quad}_{+360}432, \ldots$$



(Note: Try to convince yourself that this sequence is never Δ^k-constant *without* finding successive differences.)

In this case, we can try to find successive ratios instead:

$$\{c_n\} = \frac{1}{3}\underbrace{\quad}_{\cdot 6}2\underbrace{\quad}_{\cdot 6}12\underbrace{\quad}_{\cdot 6}72\underbrace{\quad}_{\cdot 6}432, \ldots$$

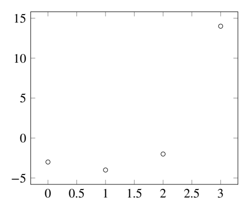
Because there is a constant ratio r from one term to the next, we say this sequence is *geometric*. Again, let's write it as a recurrence relation:

$$c_n = \begin{cases} \frac{1}{3} & \text{if } n = 0 \\ 6 \cdot a_{n-1} & \text{otherwise} \end{cases}$$

The closed formula for this sequence is $c_n = \frac{1}{3}(6)^n$.

So far we've only considered sequences whose recurrence form involves just the previous term. What if there are more terms to consider? The Fibonacci Numbers are one example, which we will look at in the exercises. Let's consider another here, along with a plot of the first few values. We will exclude the last two since they dwarf the others and obscure any interesting shape to the curve.

$$\{d_n\} = -3, -4, -2, 14, 82, 326 \ldots$$



This sequence cannot be geometric since a positive follows a negative but the sign does not simply alternate. And a short investigation shows there is no $k \in \mathbb{N}$ such that the sequence is Δ^k-constant, so it is not polynomial. With a lot of trial and error we can write it in recurrence form:

$$d_n = \begin{cases} -3 & \text{if } n = 0 \\ -4 & \text{if } n = 1 \\ 5d_{n-1} - 6d_{n-2} & \text{otherwise} \end{cases}$$

While determining the recurrence relation can be difficult, once we have it we can use *characteristic roots* to find a closed formula. We begin by rewriting the "otherwise" cases from the recurrence relation as

$$d_n - 5d_{n-1} + 6d_{n-2} = 0$$

then replace the sequence terms with powers of the variable x:

$$x^2 - 5x + 6 = 0.$$

This is called the *characteristic equation* for the sequence. If we happen to have a d_{n-3} term then we use x^3. In general, we let x^k stand in for d_n if the earliest term in the relation is d_{n-k}.

Now we solve the characteristic equation. In our case, this yields the characteristic roots $x = 2, 3$. We can then assume that a closed formula for our sequence looks like

$$d_n = A2^n + B3^n.$$

Let's solve for A and B as we did above, by examining the first couple terms.

$$\begin{aligned} d_0 &= -3 = A2^0 + B3^0 = A + B \\ d_1 &= -4 = A2^1 + B3^1 = 2A + 3B \end{aligned}$$

Solving this yields $A = -5$, $B = 2$. Therefore, we find the closed form

$$d_n = -5 \cdot 2^n + 2 \cdot 3^n.$$

As a side note, if we end up with repeated characteristic roots, i.e. if our characteristic polynomial has the form $(x-a)^k(x-b)\cdots$ for some $k > 1$, then we can use the *repeated root method* by setting our relation to

$$d_n = A_1 a^n + A_2 n a^n + \cdots + A_k n^k a^n + Bb^n + \cdots$$

So, for example, if our characteristic polynomial factors as $(x-3)^2(x+4)$ then we can write

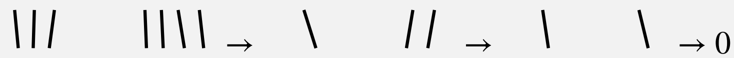
$$d_n = A_1 3^n + A_2 n 3^n + B(-4)^n$$

and solve as above.

There is another sequence type worth addressing in this text, but first we need to introduce another ruleset.

WYTHOFF NIM

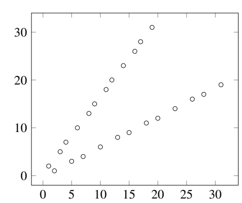
WYTHOFF NIM (also *Wythoff's Game*) is a two-heap NIM variant wherein players may remove $k > 0$ sticks from either pile or both piles at the same time.



The first player removes two sticks from both heaps. The second player then removes one to leave one in each. The first player responds by removing the sticks from both heaps at once.

The reader can verify the first $\mathcal{P}$-positions for WYTHOFF NIM, visualized as a list and plotted below. Note that if $(x, y) \in \mathcal{P}$ then $(y, x) \in \mathcal{P}$, as well.

$$(1, 2), (3, 5), (4, 7), (6, 10), (8, 13), (9, 15), (11, 18), (12, 20), (14, 23), \ldots$$



Notice that the $\mathcal{P}$-positions in two-heap NIM are $\mathcal{N}$-positions in WYTHOFF NIM. They are, in fact, positions from which the next player can win immediately. Subverting games in this way is a common method of developing new directions for mathematical study.

At first there does not seem to be much of a pattern to the $\mathcal{P}$-positions in WYTHOFF NIM. However, a closer examination of each coordinate reveals an interesting property. Below we let a_n be the smaller and b_n the larger of the coordinate pairs.

$$\begin{array}{lcccccccccccccc} a_n & = & 1 & 3 & 4 & 6 & 8 & 9 & 11 & 12 & 14 & 16 & 17 & 19 & \ldots \\ b_n & = & 2 & 5 & 7 & 10 & 13 & 15 & 18 & 20 & 23 & 26 & 28 & 31 & \ldots \end{array}$$

If we begin with $n = 1$ we can see that $a_n = \lfloor n\phi \rfloor$ and $b_n = \lfloor n\phi^2 \rfloor$, where $\lfloor x \rfloor$, the *floor* of x, is the greatest integer less than or equal to x, and $\phi = \frac{1+\sqrt{5}}{2}$ is the golden ratio we saw above. The sequences a_n and b_n are examples of *complementary Beatty sequences*. A *Beatty sequence* can be defined with any $r \in \mathbb{R}^+ \setminus \mathbb{Q}$ by

$$\mathcal{B}_r = \{\lfloor nr \rfloor\}_{n=1}^{\infty},$$

and two Beatty sequences are *complementary* if their intersection is empty and their union is $\mathbb{Z}^+$. It turns out that, for any $r \in \mathbb{R}^+ \setminus \mathbb{Q}$, the sequences $\mathcal{B}_r$ and $\mathcal{B}_s$ are complementary if $\frac{1}{r} + \frac{1}{s} = 1$.

Exercises for 0.9

0) Prove that $F(0) + F(2) + F(4) + \ldots + F(2n) = F(2n+1) - 1$ for all $n \in \mathbb{Z}^+$.

$\star$ **1)** Find a closed formula for the sequence

$$-7, -2, 3, 8, 13, 18, \ldots$$

(Answer 0.9.1 in Appendix)

2) Find a closed formula for the sequence

$$\frac{1}{3}, \frac{13}{12}, \frac{11}{6}, \frac{31}{12}, \frac{10}{3}, \ldots$$

$\star$ **3)** Find a closed formula for the sequence

$$1, 0, 3, 10, 21, 36, 55, \ldots$$

(Answer 0.9.3 in Appendix)

4) Find a closed formula for the sequence

$$-1, 1, 9, 29, 67, 129, 221, 349, \ldots$$

⋆ **5)** Find a closed formula for the sequence

$$4, 2, 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$$

(Answer 0.9.5 in Appendix)

6) Find a closed formula for the sequence

$$-2, 6, -18, 54, -162, 486, \ldots$$

⋆ **7)** Find a closed formula for the recurrence relation

$$a_n = \begin{cases} 2 & \text{if } n = 0 \\ 3 & \text{if } n = 1 \\ 2a_{n-1} - a_{n-2} & \text{otherwise} \end{cases}$$

(Answer 0.9.7 in Appendix)

8) Find a closed formula for the recurrence relation

$$a_n = \begin{cases} 1 & \text{if } n = 0 \\ 1 & \text{if } n = 1 \\ 0 & \text{if } n = 2 \\ 6a_{n-1} - 11a_{n-2} + 6a_{n-3} & \text{otherwise} \end{cases}$$

9) Find a closed formula for the Fibonacci sequence

$$F_n = \begin{cases} 1 & n = 0 \text{ or } 1 \\ F_{n-1} + F_{n-2} & \text{otherwise} \end{cases}$$

⋆ **10)** List the first 8 terms in the Beatty sequence $\mathcal{B}_\pi$. (Answer 0.9.10 in Appendix)

11) List the first 8 terms in the Beatty sequence $\mathcal{B}_{\sqrt{2}}$.

⋆ **12)** Consider the Beatty sequence

$$\mathcal{B}_{\sqrt{3}} = 1, 3, 5, 6, 8, 10, 12, 13, \ldots$$

Find the first 5 terms of its complementary sequence $\mathcal{B}_s$, and then determine the appropriate value for s. (Answer 0.9.12 in Appendix)

13) Consider the Beatty sequence

$$\mathcal{B}_{\sqrt{5}} = 2, 4, 6, 8, 11, 13, 15, 17, \ldots$$

Find the first 5 terms of its complementary sequence $\mathcal{B}_s$, and then determine the appropriate value for s.

14) Determine the game value of the WYTHOFF NIM positions $(1, 1), (2, 2)$, and $(3, 3)$.

⋆ **15)** Come up with a sequence such that successive terms have ratios that themselves have a fixed ratio between them (i.e. like a Δ^2-constant sequence but with ratios instead of differences). (Answer 0.9.15 in Appendix)

1. Impartial graph games

Since graphs are such important structures in the study of discrete mathematics, it's no surprise that there is a lot of active work in games related to graphs. We've already seen one such game, NODE KAYLES. In this chapter, we will look at a few more of them (there are lots!), their mathematical properties, and how they can inform our study of graphs in general.

1.1. Geography

On the road again
Goin' places that I've never been
Seein' things that I may never see again
And I can't wait to get on the road again

- Willie Nelson , "On the road again "

Let's examine a game that many readers have likely played before. It's a common road trip game since it doesn't require a board and can be played through conversation.

GEOGRAPHY

GEOGRAPHY is an impartial ruleset in which players take turns naming towns, cities, or countries under the restrictions that no place may be named twice, and the next place named must begin with the last letter of the previously name place. "House" rules differ, sometimes allowing only cities or only countries to be named.

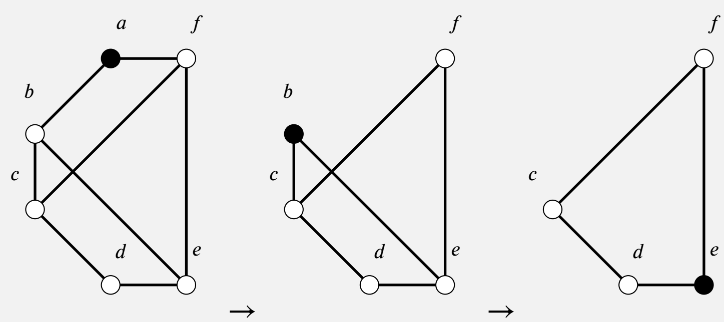
AntwerP → PlymoutH → HalifaX → XanadU

Beginning with Antwerp, subsequently named cities begin with the ending letter of the previous play.

While it's a fun game, there is not much analysis that we can do on GEOGRAPHY as defined. There are too many variations and too many ways to spell transliterated places named throughout the world. However, what we can do is generalize the game into something that we can analyze.

UNDIRECTED VERTEX GEOGRAPHY

UNDIRECTED VERTEX GEOGRAPHY is an impartial game on an undirected graph, wherein players take turns choosing a neighbor of the previously chosen vertex, then deleting that previous vertex from the graph.

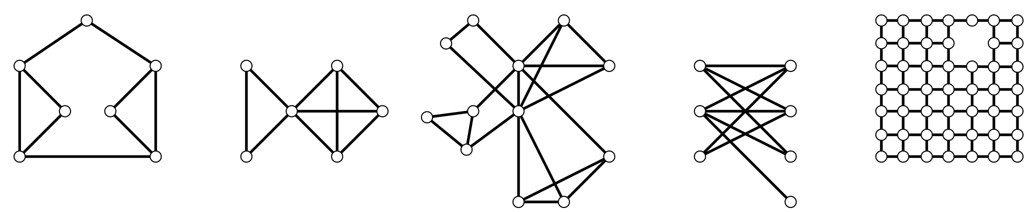


Starting at vertex a, the first player moves to b and deletes a. The next player then moves to e and deletes vertex b.

In Undirected Vertex Geography, we always have to keep track of which vertex is active at any given time. So a position in the game consists of a graph and a token on one of the graph's vertices. You may notice that Undirected Vertex Geography doesn't quite capture what's going in in Geography, since using undirected edges implies that there is no particular direction to the relationship between two nodes. Whereas we know that, in Geography, there is a move from Plymouth to Halifax but not the other way. But we begin with the undirected case, and will address the directed case later on.

Undirected Vertex Geography is a graph game, which means that we can spend some time playing around with graphs. That means asking CG-type questions, as well as questions about graphs in general. For example, let's examine which nimbers are achievable by positions in the game. The ruleset has Nim *dimension n* if $*2^{n-1}$ is achievable but $*2^n$ is not.

Note that not only can we find an Undirected Vertex Geography position associated with every possible nimber, demonstrating that the ruleset has infinite Nim dimension, but that these are achievable by restricting ourselves to trees. In fact, Undirected Vertex Geography starting at the root of a tree is equivalent to Tree. Let's look next at play on some graphs with cycles.



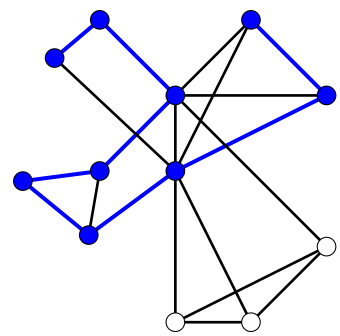
G H F J K

What is the longest possible game on each of the graphs above? You can choose to begin anywhere for your first move. Let's look at the graphs one at a time.

In G, no matter where we begin we see that we can hit every vertex exactly once. So a game on G can last for as many turns as the order of the graph. In fact, G has an even stronger property of containing a *hamiltonian cycle*, a cycle starting and ending at any single vertex in which every vertex is used exactly once. We say that a graph with a hamiltonian cycle is a *hamiltonian graph*.

Try to find a hamiltonian cycle in H. Give up? You probably noticed why H is not hamiltonian: it has a *cut vertex*. In other words, there is a vertex that, when removed, leaves at least two connected components. No graph with a cut vertex is hamiltonian because any path through this vertex cannot return to the start without using the cut vertex again. However, H does have a path that starts and ends on different vertices and includes all vertices of the graph, called a hamiltonian *path*, making H *traceable*. So a game of UNDIRECTED VERTEX GEOGRAPHY can last as many turns as the order of H, as well.

The graph F has no cut vertices so F is *two connected*, though we do see a pair of vertices that, if removed, separate the graph. We say that the *connectivity* of H is two, written $k(H) = 2$, since there are two vertices that, when removed, disconnect the graph. Is F hamiltonian or traceable?



Since any hamiltonian path through F would have to include each of the two cut vertices exactly once, but there are four components separated by these two vertices, there is no way to reach all of them in a single hamiltonian cycle, let alone a hamiltonian path. So F is not traceable.

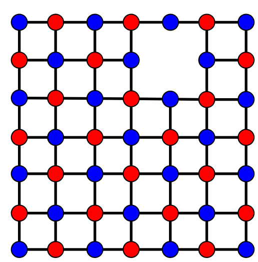
It can be quite difficult to determine whether or not a graph is hamiltonian, but there are some results that we can depend on.

Theorem 1.1.1. *If G is a connected graph with cut set of size n which separates G into at least $n + 1$ connected components, then G is not hamiltonian.*

Proof. Label the cut set $S = \{v_0, v_1, \dots, v_{n-1}\}$ and let $U = \{u_1, u_2, \dots, u_n\}$ be vertices such that u_i is in component i after removing all the vertices in S from the graph G. Let C be a longest possible cycle in G. Any traversal of C that includes two vertices from U must include at least one element from S between them. There is no cycle in G that includes more than $|S| = n$ elements from U, and hence not every vertex in U can be on the cycle. Therefore, G is not hamiltonian. □

Graph J is an example of a *bipartite graph*. Its vertices can be partitioned into two sets X and Y, each of which has no edges, while some edges may exist between X and Y. In a bipartite graph, all edges adjoin vertices in different partite sets. We can employ Theorem 1.1.1 to see that, by removing all vertices on the left, we are left with four isolated vertices. So there is no hamiltonian cycle in J. However, there *is* a hamiltonian path. So J is traceable, and a game of UNDIRECTED VERTEX GEOGRAPHY can last for seven turns.

Finally, look at graph K. We can try to find a hamiltonian path exhaustively, through trial and error, or we could look carefully at the properties of the graph. Note that K is a 7×7 vertex grid with a single vertex removed, and a grid graph is bipartite. You can see this by coloring alternating vertices, effectively partitioning the vertices into two sets.



Our graph K is a 7×7 grid graph with a vertex removed. If we partition the vertices, then we see that the missing vertex *would* be colored red. This demonstrates that K is a bipartite graph with one set of size 25 and the other of size 23. Not only does K not have a hamiltonian cycle by Theorem 1.1.1, but, as you will see in the exercises, it is also not traceable.

Now, as with all impartial combinatorial games, we'd like to characterize the $\mathcal{P}$-positions. Thankfully this is not too difficult for UNDIRECTED VERTEX GEOGRAPHY, at least in the undirected case.

Math Diversion: Matchings

A *matching* in a graph G is a collection of edges with distinct endpoints. Think of a pairing of people in your class. A matching *saturates* the endpoints of its edges, and is is *perfect* if it saturates every edge of the graph. For example, in the even cycle C_{10} with vertices $\{v_0, v_1, \ldots, v_9\}$ in which each vertex v_i has as its neighbors the vertices v_{i-1} and v_{i+1}, under arithmetic mod 10, there is a perfect matching given by the set of edges

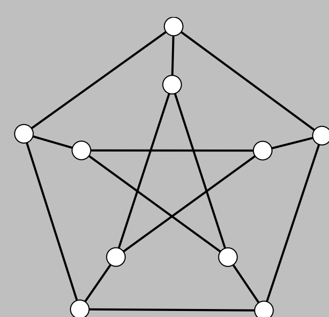
$$\{v_0v_1, v_2v_3, \ldots, v_8v_9\} = \{v_{2k}v_{2k+1} | 0 \leq k \leq 4\}$$

as well as the perfect matching

$$\{v_9v_0, v_1v_2, \ldots, v_7v_8\} = \{v_{2k-1}v_{2k} | 0 \leq k \leq 4\},$$

but odd cycles do not have perfect matchings.

Consider the Petersen Graph below. How many perfect matchings can you find?



A.: *There are six perfect matching in the graph.*

We say that a matching M is *maximal* if it is not contained in a larger matching. In other words, M is maximal if there is no edge outside of M that doesn't share one of its vertices with an edge in M. Try to find a graph with a maxim*al* matching that is not a maxim*um* matching.

> **A:** *A path graph P_{2k+1} on $2k+1$ vertices for any $k \in \mathbb{Z}^+$ works here. There is a perfect matching in the graph, but also a maximal matching excluding the two endpoints.*

We can use matchings to determine the $\mathcal{P}$-positions in UNDIRECTED VERTEX GEOGRAPHY.

Theorem 1.1.2. *An* UNDIRECTED VERTEX GEOGRAPHY *position* (G, v) *on graph* G *with token on vertex* v *is in* $\mathcal{N}$ *if and only if every maximum matching saturates* v.

Proof. First, assume that every maximum matching saturates v. We will demonstrate that $(G, v) \in \mathcal{N}$. The first player can simply choose an edge in the maximum matching. The other player is forced to use an edge outside this matching, to which the first player can, again, play along an edge in the matching, and so on. If no such edge exists for the first player at any point then there must have been another matching of the same size that did not saturate vertex v. So we have found a strategy for the first player to win, and $(G, v) \in \mathcal{N}$.

Mirroring the above, if there is a maximum matching in G that does not saturate v, then there must be a winning strategy for the second player always using edges in this maximal matching. Hence $(G, v) \in \mathcal{P}$. □

Exercises for 1.1

⋆ **0)** Find the shortest possible game of UNDIRECTED VERTEX GEOGRAPHY in the graph G above. (Answer 1.1.0 in Appendix)

1) Find the shortest possible game of UNDIRECTED VERTEX GEOGRAPHY in the graph H above.

2) Find the shortest possible game of UNDIRECTED VERTEX GEOGRAPHY in the graph F above.

⋆ **3)** Find the shortest possible game of UNDIRECTED VERTEX GEOGRAPHY in the graph J above. (Answer 1.1.3 in Appendix)

4) Find the shortest possible game of UNDIRECTED VERTEX GEOGRAPHY in the graph K above.

⋆ **5)** Is the Petersen Graph hamiltonian? Is it traceable? (Answer 1.1.5 in Appendix)

6) Remove any single vertex from the Petersen Graph. Is the resulting graph hamiltonian? Is it traceable?

7) Research the Petersen graph and find one interesting fact about it

⋆ **8)** Prove that every tree is bipartite. (Answer 1.1.8 in Appendix)

9) Prove that any bipartite graph G with parts X and Y, $|X| < |Y| - 1$ is not traceable.

⋆ **10)** Let C_n be the cycle graph on n vertices (drawn as an n-gon). Find a formula for the size $L(C_n)$ of the largest possible matching in C_n, and then find a formula for the size $S(C_n)$ of the smallest possible maximal matching in C_n, in terms of n. (Answer 1.1.10 in Appendix)

11) Let Pi_n be the pinwheel graph on n vertices (drawn as single vertex with $n-1$ neighbors forming triangles in pairs, with potentially one pendant edge). Find a formula for the size $L(Pi_n)$ of the largest possible matching in Pi_n, and then find a formula for the size $S(Pi_n)$ of the smallest possible maximal matching in Pi_n, in terms of n.

1.2. Undirected Edge Geography

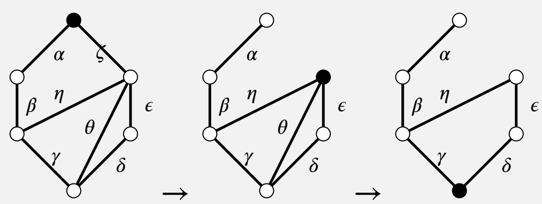
I've been everywhere, man
I've been everywhere, man
Across the deserts bare, man
I've breathed the mountain air, man
Of travel I've had my share, man
I've been everywhere

- Geoff Mack , "I've been everywhere "

Next we consider another variant of GEOGRAPHY.

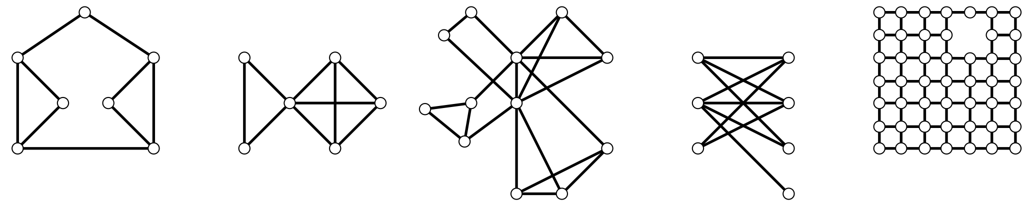
UNDIRECTED EDGE GEOGRAPHY

UNDIRECTED EDGE GEOGRAPHY is played similarly to UNDIRECTED VERTEX GEOGRAPHY, but with the modification that once an edge is played, only that edge is removed from play. Vertices remain.



The first player moves along edge ζ. The second player responds by moving along edge θ.

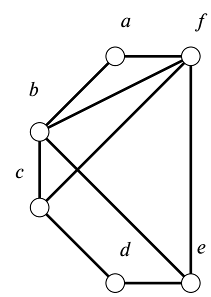
In UNDIRECTED EDGE GEOGRAPHY, players remove edges instead of vertices as play progresses. This naturally leads to the same question we answered regarding UNDIRECTED VERTEX GEOGRAPHY: What is the longest possible game? To answer this, let's return to the same graphs we looked at last time.



G *H* *F* *J* *K*

For each graph, try to start at a single vertex and generate a tour using every edge once and only once. This is called an *Euler Tour*. Any Euler Tour that starts and ends at the same vertex is an *Euler Circuit*.

You'll notice that this is impossible for any of the graphs shown. Any vertex with an odd number of neighbors, i.e. a vertex with odd degree, must be the first or last vertex in this sort of tour, and each of the graphs shown have more than two vertices with odd degree. But is this necessary condition of having no more than two vertices of odd degree sufficient for a graph to have an Euler Tour? Let's instead examine another graph with fewer vertices of odd degree. Call the graph below X.



Try tracing out an Euler Tour in X. You now know that, if it exists, it must begin and end at the vertices c and e, since these are the only vertices of odd degree. And you should see that, yes, it's possible. Let's talk a bit about Euler Circuits before addressing the sufficiency question. What must be true about a graph for it to contain an Euler Circuit?

Similar to the necessary condition for a graph to have an Euler Tour, any graph with an Euler Tour must have all even degrees. In fact, this condition is both *necessary* and *sufficient*.

Theorem 1.2.1. *A connected graph G has an Euler Circuit if and only if all its degrees are even.*

Proof. We must prove both directions. First, any circuit contributes an even number of adjacencies to each vertex in the circuit, as any traversal along the circuit must both enter and exit each vertex at each visit. So it is impossible for a graph with any vertices of odd degree to contain an Euler Circuit.

Next, assume the connected graph G has vertices of only even degree. Label the vertices $\{v_0, v_1, \ldots, v_n\}$ and their respective degrees $\{d(v_0), d(v_1), \ldots, d(v_n)\}$. Beginning at any vertex, let C be a circuits containing the vertices $\{v_{c_0}, v_{c_1}, \ldots, v_{c_k}\}$. Remove the edges from C from the graph G and what remains is another graph on the same vertices with only even degrees. Repeating this process until we have removed all edges from G, we end up with a collection of circuits $\{C_0, C_1, \ldots, C_j\}$. Any two circuits sharing at least one vertex can be stitched together in a natural way to result in a single circuit. If we repeat *this* process as many times as possible, then we will end up with a single circuit that includes every edge in G exactly once. □

Determining whether or not a graph has en Euler Circuit is *much* easier than determining whether or not it has a Hamiltonian Cycle. We need only examine the degrees. Note that there was no requirement within our proof that the graph be simple. I.e., any connected graph with all even degrees, whether or not it contains multiple edges, has an Euler Circuit. Let's return to the graph X above. We know from Theorem 1.2.1 that X does *not* contain an Euler Circuit, but it does contain an Euler Tour. In fact, we can show that having exactly two vertices of odd degree is both a necessary and a sufficient condition for a graph to have an Euler Tour.

Theorem 1.2.2. *A connected graph G has an Euler Tour if and only if it has exactly two vertices of odd degree.*

Proof. No graph with more than two vertices of odd degree can have an Euler Tour by our argument above, and the endpoints of an Euler tour must have odd degree. If G is a graph with two vertices, u and v, with odd degree, then consider the graph G' composed of a copy f G with the addition of the edge uv. By Theorem 1.2.1, the graph G' has an Euler Circuit. This circuit, minus the additional edge uv, is an Euler Tour in the graph G. □

In order to prove Theorem 1.2.2 we needed the flexibility to include any type of graph, even one with multiple edges, in Theorem 1.2.1.

Exercises for 1.2

⋆ **0)** What, if any, are the restrictions on n and m such that $K_{m,n}$ has (i) an euler circuit and (ii) an euler tour? (Answer 1.2.0 in Appendix)

1) For which values of n do K_n, W_n, $K_{n,n}$, C_n, and Q_n have euler circuits? Note that Q_n is the graph on 2^n vertices where each vertex is a bit string of length n, and an edge exists between any two vertices whose bit strings differ in exactly one position (e.g. 1101 is adjacent to 0101, 1001, 1111, and 1100).

2) What is the fewest number of edges we need to add to the Petersen Graph to give it (i) an euler circuit, and (ii) an euler tour? Draw the appropriate graphs.

1.3. Directed Geography

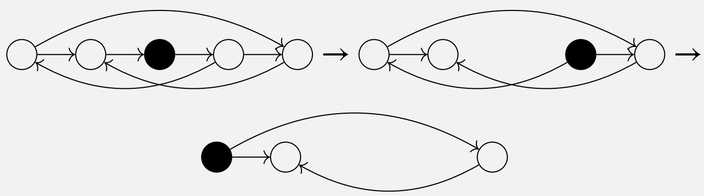
We round out this chapter on graph games by returning to GEOGRAPHY as defined earlier. If every place on earth (and beyond) is the label of a distinct node in a directed graph G, and we add a directed edge from one node to another whenever the associated labels match the alphabetical requirements of GEOGRAPHY, then G along with a token is a position in DIRECTED GEOGRAPHY. There is one caveat, however. Note that G as we have defined it has cycles, like

$$\text{Richmond} \rightleftharpoons \text{Denver.}$$

We need to make sure to remove each vertex as its played, just as in UNDIRECTED VERTEX GEOGRAPHY.

DIRECTED GEOGRAPHY

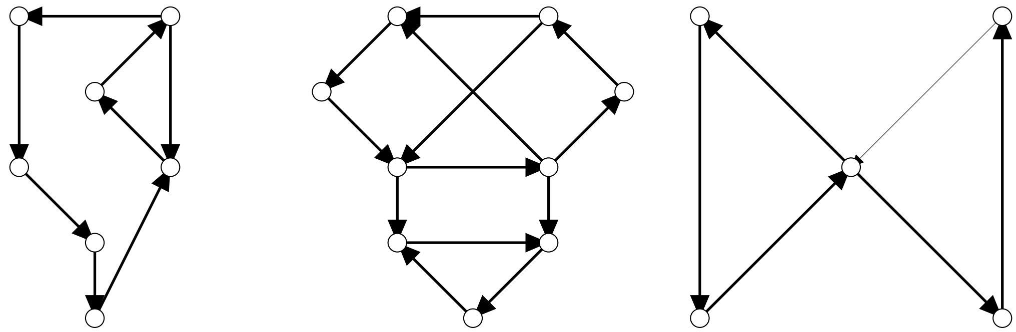
DIRECTED GEOGRAPHY is an impartial game on an undirected graph, and a token on one of the vertices. A turn consists of moving the token along an outgoing edge to a new vertex, then removing the prior vertex from the graph. DIRECTED GEOGRAPHY was first known as GENEREALIZED GEOGRAPHY, as it is a generalization of GEOGRAPHY to graphs. It is also known as DIRECTED VERTEX GEOGRAPHY and often colloquially just as GEOGRAPHY.



The token begins on the middle vertex. The first player moves it to the right, deleting the middle vertex they came from. The second player then uses the edge heading back to the left side.

Every impartial game graph we've seen in the book is actually a game of DIRECTED GEOGRAPHY! As you move a token in this game, you can imagine traversing the game graph of any impartial game, until neither player has a move remaining. Therefore, if we happen to find a nice, quick way to solve every game of DIRECTED GEOGRAPHY, then we can solve *every* impartial game just as easily!

Consider a directed graph G. Just like the hamiltonian paths and cycles we found while studying UNDIRECTED VERTEX GEOGRAPHY, we have hamiltonian paths and cycles in directed graphs. Directed graphs have a new property, though, that we haven't seen in undirected graphs: *strong connectedness*. A directed graph (digraph) is *strongly connected* if, for every pair of vertices u, v, there is a directed path from u to v and a directed path from v to u. Consider the three graphs below.



D *G* *H*

Digraph D is strongly connected. We can see this because D has a directed hamiltonian cycle. So, just as in UNDIRECTED VERTEX GEOGRAPHY, a game of DIRECTED GEOGRAPHY could include every node, no matter which node the game begins on.

What about G? You'll notice that there is no way to get from the bottom vertex to either vertex at the top of the digraph, so G is not strongly connected. The digraph is composed of two strongly connected components (or *strong components*): the component at the top is a six-cycle with a couple more arcs, and the component at the bottom is a directed three-cycle. These two components are joined by a pair of arcs, both oriented from the first to the second. So, in fact, no path joins a vertex in the second strong component to a vertex in the first. However, choose any pair of vertices in G and label them u and v. There is a path from u to v *or* a path from v to u. Although G is not strongly connected, it is *weakly connected*.

Finally, let's examine the digraph H. There is no hamiltonian cycle in H, which we know by Theorem 1.1.1. However, there *is* a directed hamiltonian path. Furthermore, H is also strongly connected because for every pair of vertices u and v, there is both a $u-v$ path and a $v-u$ path in H. It's worth noting that having a hamiltonian cycle, having a hamiltonian path, and being strongly connected are all distinct properties. While being hamiltonian implies that a digraph is both strongly connected and traceable, there are no other assumptions we can make about these three digraph properties.

Exercises for 1.3

⋆ **0)** Prove that if a digraph is strongly connected then it is also weakly connected. (Answer 1.3.0 in Appendix)

1) What is the greatest possible number of turns in a game of DIRECTED GEOGRAPHY on graph *G* above?

2) What is the shortest possible game of DIRECTED GEOGRAPHY on graph *G* above?

3) Sketch a weakly connected digraph with four strong components.

⋆ **4)** Consider the game of GEOGRAPHY played on Canadian Provinces and Territories. Draw the corresponding DIRECTED GEOGRAPHY graph. Label your vertices with the two-character shorthands. You don't know what those are, eh? Here's the list: Alberta (AB), British Columbia (BC), Manitoba (MB), New Brunswick (NB), Newfoundland and Labrador (NL), Nova Scota (NS), Northwest Territories (NT), Nunavut (NU), Ontario (ON), Prince Edward Island (PE), Quebec (QC), Saskatchewan (SK), Yukon (YT). (Answer 1.3.4 in Appendix)

5) Programming Question: Let a GEOGRAPHY position be given by a pair, a list of strings, then a single string inside that list. Write a function `get_options` that takes a position like that and returns the list of options from that position. (These options should each be a pair of the same form.)

For example, if position `G = (['ogunquit', 'saco', 'orlando'], 'saco')`, then
`get_options(G)` will return a list with two options:

```
[(['ogunquit', 'orlando'], 'ogunquit'),
 (['ogunquit', 'orlando'], 'orlando')]
```

6) Use your answer from exercise 1.3.5 to write a function `print_geography_nimbers` that takes a list of strings and prints out the nimber of each GEOGRAPHY position starting from each string in the list. (You can reuse the `mex` function we wrote previously, and you'll probably want to write a `get_nimber(geography)` function.)

Here's a sample case using the example above (in Python interactive mode):

```
»> print_geography_nimbers(['ogunquit', 'saco', 'orlando'])
(['ogunquit', 'saco', 'orlando'], 'ogunquit'):  *0
(['ogunquit', 'saco', 'orlando'], 'saco'):  *2
(['ogunquit', 'saco', 'orlando'], 'orlando'):  *1
```

7) Prove the following statement: For any $n \in \mathbb{N}$ where $n \geq 1$, there exists a directed graph with n nodes such that a DIRECTED GEOGRAPHY position starting on one vertex from that graph, G, is equal to $*(n-1)$. (Hint: you can use strong induction to prove this constructively by creating a working graph.)

2. Partisan Games

Tick Tac Toe
no two turns in a row:
that's cheating

- Lisa Loeb, "Sick, Sick, Sick"

[1]

So far we have only studied impartial rulesets, where both players always have the same moves. Most combinatorial games that people are familiar with don't fall into that category. For example, in CHECKERS (or DRAUGHTS[2]) each player may only move pieces of their own color. (Black for L and red for R.) For the remainder of this text, we will be learning how to evaluate *partisan games*: rulesets where the players may have different move options.

Since there are many two-player games that use two colors, there is a convention for which color "belongs" to each player:

L	R
bLue	Red
bLack	White
Positive	Negative
True	False

Sometimes the players are named Louise and Richard[3] for L and R, respec-

[1] https://www.lisaloeb.com/sick-sick-sick

[2] The authors of this text are from the U.S., so we know the game as "Checkers".

[3] Named after Louise and Richard Guy. Richard was one of the three authors of the text *Winning Ways for your Mathematical Plays*. Along with coauthors John H. Conway and Elwyn Berlekamp, they introduced the notation we use for partisan games.

tively. Because of this, in many published CGT articles and books, you may see L referred to with feminine pronouns and R with masculine pronouns.

Sometimes we need another color to refer to game pieces or markers that either player can use for their move. These are usually either given a Green or Gray color.

2.1. Partisan Game Notation

For impartial game positions, we used this notation: $G = {}^{*}\left\{\, 0, *, *3, *6 \,\right\}$. For partisan positions, we will need to separate the moves into two sections: the options for each of the two players. Our notation, known as *option notation* will use the same braces as for sets, but will be divided into two "sides", one each for L and R.

In option notation, every partisan game position can be written as: $\{\ L\text{'s options} \mid R\text{'s options}\ \}$. We can consider any impartial game to also be partisan, so we could rewrite the impartial position above:

$$ {}^{*}\left\{\, 0, *, *3, *6 \,\right\} = \left\{\, 0, *, *3, *6 \,\middle|\, 0, *, *3, *6 \,\right\} $$

Some common values we'll see are: $0 = {}^{*}\{\ \} = \{\ \mid\ \}$ and $* = {}^{*}\left\{\, 0 \,\right\} = \left\{\, 0 \,\middle|\, 0 \,\right\}$. This last one, for example, is colloquially pronunced "zero slash zero". From this point on, we will forego the impartial notation and focus on option notation[4].

Sometimes we can simplify a partisan position to a single nimber, even if the two sides don't have exactly the same nimbers on each side. For example:

$$ \left\{\, 0, *2 \,\middle|\, 0, *3 \,\right\} = * $$

This position resolves to $*$ because the mex of the nimbers of both sides is the same. We can check our work by showing that $\left\{\, 0, *2 \,\middle|\, 0, *3 \,\right\} + * \in \mathcal{P}$:

- If L goes first, and chooses the zero option of either component, then R can respond by moving to the zero option of the other. If L instead chooses

[4]The impartial notation we have used is not standard in CGT.

to move the first component to $*2$, then R can choose the $*$ option from that, returning the position to 0.

- If R goes first, L will have the same choices to return the sum to a zero position. If R chooses the zero from either component, then L can choose the zero from the other component. If R instead picks the $*3$ option from the first component, L can choose the $*$ option from $*3$, moving the sum to zero again.

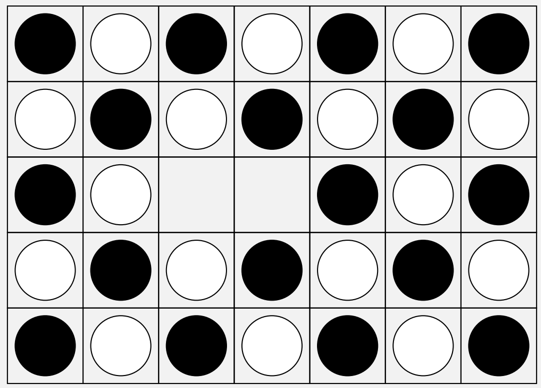
Notice that unlike impartial games, to show that a position is in $\mathcal{P}$, we have to show that neither of the players has a winning move option. If they have different options, then we have to argue none of them can win by considering the two separate cases where they each go first, as above.

If the two sides contain only nimbers and both sides have the same mex of those nimbers, then the value is the nimber of that common mex. However, if the two sides don't have the same mex, then we cannot simplify the result into a single nimber. For example, there is no simpler notation for $\{\, 0 \mid 0, * \,\}$.

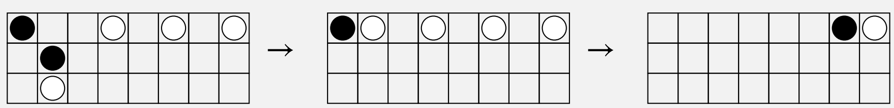
Nimbers appear in many partisan games, especially $*$. One easy example occurs in the game KONANE.

KONANE

KONANE is a traditional Hawaiian game that is similar in many ways to Checkers or Draughts. Black and white stones begin on a grid in an alternating fashion, with two adjacent stones removed:

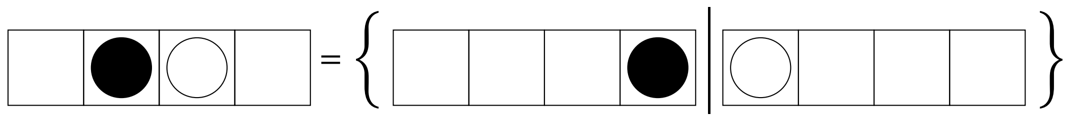


Each turn, the current player uses one of their stones to jump over an opponent's orthogonally-adjacent piece to the empty spot on the other side. If there is another opponent piece (and empty space behind) along the same line, the jumping can continue if the player wishes.
GENERALIZED KONANE is a variant where starting stones don't need to alternate colors. In other words, two stones of the same color can be orthogonally adjacent.



R makes the first move by jumping their lower piece up. This sets up a big move for *L*, who makes a triple jump.

Let's use option notation to find and simplify the value of a KONANE position:



$$= \{ 0 \mid 0 \}$$
$$= *$$

The exercises below include some more KONANE problems.

Exercises for 2.1

⋆ **0)** Rewrite $^*\{ 0, *, *5, *7 \}$ in option notation. Do not simplify to a single nimber value. (Answer 2.1.0 in Appendix)

1) Rewrite $^*\{ 0, *9, *10 \}$ in option notation. Do not simplify to a single nimber value.

⋆ **2)** Simplify $\{ 0, *, *2, *4 \mid 0, *, *2, *4 \}$ to a single nimber. (Answer 2.1.2 in Appendix)

3) Simplify $\{ *, *2, *3, *4, *5 \mid *, *2, *3, *4, *5 \}$ to a single nimber.

⋆ **4)** Can we simplify $\{ 0, *, *2, *3 \mid 0, *, *2 \}$ to a single nimber? If so, provide that nimber value. (Answer 2.1.4 in Appendix)

⋆ **5)** Can we simplify $\{ 0, *, *2, *4 \mid 0, *, *2, *5, *6 \}$ to a single nimber? If so, provide that nimber value. (Answer 2.1.5 in Appendix)

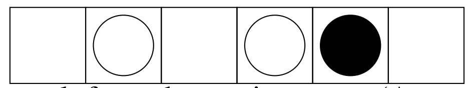
6) Can we simplify $\{ 0, *, *3, *5, *7, *9 \mid 0, *, *3, *4, *5, *6 \}$ to a single nimber? If so, provide that nimber value.

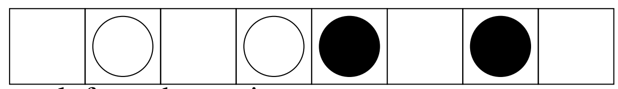
7) Can we simplify $\{ 0, *, *2, 3, *4 \mid *, *2, *3 \}$ to a single nimber? If so, provide that nimber value.

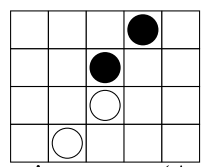
⋆ **8)** Can we simplify $\left\{ *, *2, *4, *5 \,\middle|\, *2, *4, \{ 0 \mid 0 \} \right\}$ to a single nimber? If so, provide that nimber value. Show your work. (Answer 2.1.8 in Appendix)

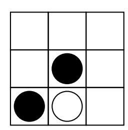
9) Can we simplify $G = \left\{ \{\ \mid\ \}, \{ 0 \mid 0 \} \,\middle|\, \{ \{\ \mid\ \} \mid 0 \}, \{\ \mid\ \} \right\}$ to a single nimber? If so, provide that nimber value. Show your work.

10) Can we simplify $G = \left\{ \{\ \mid\ \}, \{ 0 \mid 0 \}, *4, *5 \,\middle|\, \{ 0, *, *2 \mid *, 0, *5\ *2 \}, *5, 0, * \right\}$ to a single nimber? If so, provide that nimber value. Show your work.

⋆ **11)** What is the value of the KONANE position  ? You can simplify your analysis by using the result from the section text. (Answer 2.1.11 in Appendix)

12) What is the value of the KONANE position  ? You can simplify your analysis by using the result from the section text.

⋆ **13)** What is the value of the KONANE position  ? You can simplify your analysis by using the result from the section text. (Answer 2.1.13 in Appendix)

14) What is the value of the KONANE position  ?

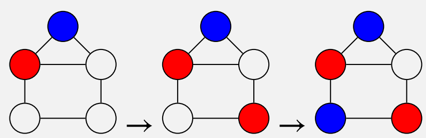
15) Write a function, `is_nimber(left_nimbers, right_nimbers)` that returns whether a game position with only those two lists of nimbers is itself a nimber.

2.2. Game Trees and Outcome Classes

Let's look at another game where the players have different moves.

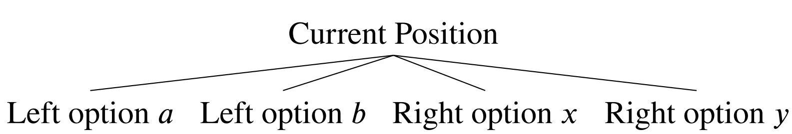
COL

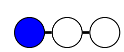
COL is a game played on a graph. L uses the color Blue, while R uses Red. Each turn, a player chooses an uncolored vertex, v, that is not adjacent to any vertices in their color, then paints v with their color.

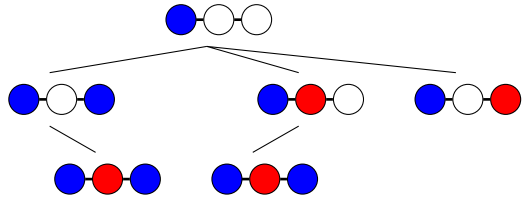


R makes the first move, choosing the only vertex they can to color red. After that, L also has only one move, so they color that vertex blue.

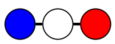
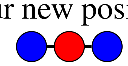
The big aspect to take note of off the bat is that each player is restricted to only one color. L colors vertices Blue (and cannot use Red) and R can likewise only use Red. Since these moves are restricted to only one player, we want to represent that access in our game trees as well as our formal notation. To do this, our partisan game trees will look quite different from the impartial ones. Instead of showing children with connecting lines that only travel parallel to the x and y axes, we will draw straight lines. Those extending down and to the left indicate options only for L; those extending down and right indicate options only for R.

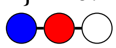
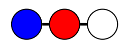
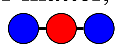


Let's jump into an example and draw the entire tree for COL on:  :

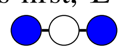


Just as with impartial games, this game tree gives us the full picture of what can happen from the initial position. An energetic student might want to immediately start evaluating this to determine the value or outcome class. Unfortunately, our old tools aren't quite enough this time.

Some of our new positions have values we've already seen. For example, both  and  have no moves available. They are both equal to $\{\ |\ \} = {}^*\{\ \} = 0$.

 is another matter, however.  $= \left\{ 0 \,\middle|\, \right\}$, because the left player can move to , which, as we said, is equal to 0. This doesn't fit nicely into our outcome classes $\mathcal{N}$ and $\mathcal{P}$. Here:

- The next player wins if they are L, but loses if they are R.
- The previous player wins if they are L, but loses if they are R.

In other words, no matter which player goes first, L has a winning strategy. The same thing could happen for R (e.g. on ). To handle these two new cases, we'll need some new outcome classes for partisan games!

- $\mathcal{L}$, "Positive": the set of positions where L has a winning strategy no matter who goes first.
- $\mathcal{R}$, "Negative": the set of positions where R has a winning strategy no matter who goes first.

In impartial games, we didn't need these two because they couldn't happen. If L could win by going first, then R could use that same strategy if they went first. Thus, impartial positions are always in $\mathcal{N}$ or $\mathcal{P}$.

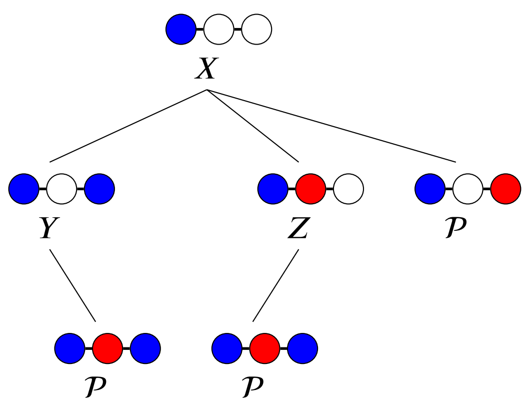
To say this another way, we can lay out all of our outcome classes in this table:

		R, Playing First	
	$o(G)$	Wins	Loses
L First	Wins	$\mathcal{N}$	$\mathcal{L}$
	Loses	$\mathcal{R}$	$\mathcal{P}$

Thankfully, we can still determine the outcome class of a game based recursively on the outcome classes of the options. For each player, we are looking for a winning move.

- L has a winning move if and only if they can move to a $\mathcal{P}$ or $\mathcal{L}$ option.
- R has a winning move if and only if they can move to a $\mathcal{P}$ or $\mathcal{R}$ option.

How can we label our vertices? We are still going to use the basic idea from doing this with impartial games, except that each level is a bit more complicated. Let's label the outcome classses on our tree above as an example. First, just as before, we label the terminal nodes as zero. From there we'll determine the other outcome classes, which I've labelled with X, Y, and Z.

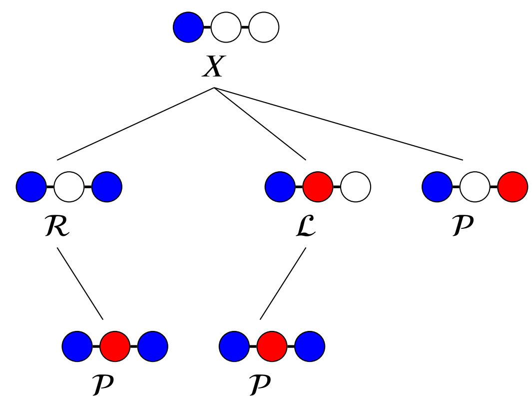


To figure out the outcome class of Z, $o(Z)$, we look at the options and see which players have a winning move.

- L has a move (their only move), to a $\mathcal{P}$-position, so they do have a winning move. Thus, we've narrowed down Z's outcome class to either $\mathcal{N}$ or $\mathcal{L}$.

- R doesn't have any moves, so they don't have a winning move. Thus, the outcome class is not $\mathcal{N}$ and must be $\mathcal{L}$.

To find $o(Y)$, we see that L doesn't have any moves, but R does have a winning move, so $o(Y) = \mathcal{R}$. Before we find the outcome class of the root position, let's update the tree:



At the root, we have one move for L and two moves for R.

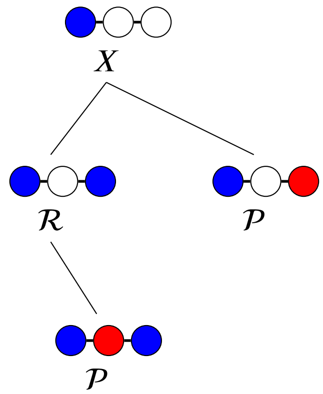
- L has only a move to $\mathcal{R}$, which is not a winning move. L does not have a winning move, so the overall outcome class is either in $\mathcal{R}$ (if R does have a winning move) or $\mathcal{P}$ (if R doesn't have a winning move).

- R has a move to $\mathcal{L}$ (not winning) and a move to $\mathcal{P}$ (a winning move). R does have a winning move, (that second move), so $o(X) = \mathcal{R}$.

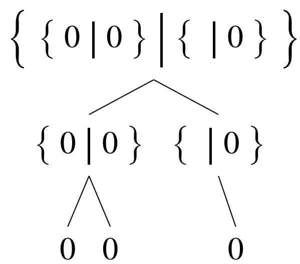
R has a winning move and L does not, so the overall outcome class is $\mathcal{R}$. Drawing the game tree and labelling the positions with outcome classes is a fine proof of this.

Just as with impartial games, in many cases we don't have to draw out the entire tree. If we are evaluating options, as soon as we find a winning one for one player, we no longer need to analyze the other options for that player. To prove that there are no winning options for a player, however, we need to analyze all of them to demonstrate that. A trimmed game tree is still a working proof of the outcome class of a position.

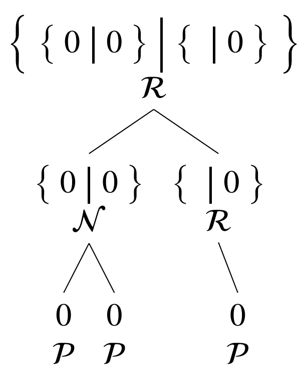
In our example, we can simplify things a bit by removing one of R's options:



We can transform back and forth between games written out formally and written out as a game tree. Often it is hard to visualize the notation and a game tree can really help out. For example, if we are looking to find the outcome class of $\left\{ \{ 0 \mid 0 \} \,\middle|\, \{ \mid 0 \} \right\}$, it's probably easier to first look at this in tree form:



Now we can label the positions and get the overall outcome class:



Computational Corner: Partisan Outcome Classes

To determine between the four outcome classes, we'll have to use separate lists for the left and right options. We might want to write a function to do this:

```
>>> get_outcome(['R', 'P', 'L'], ['N', 'L', 'N'])
'L'
```

Inside we'll need to determine each player's winnability separately. Let's start by writing a function to determine whether left has a winning move. We're going to do a cute thing with Python sets. If L has a winning option, we'll return the set with `'L'` and `'N'`. Otherwise, we'll return the set with `'R'` and `'P'`.

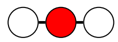
```
def get_outcomes_by_left(left_outcomes):
    '''Returns the possible overall outcomes depending on whether
Left can win.'''
    outcomes = set()
    for outcome in left_outcomes:
        if outcome in 'PL':
            # we found a winning move!
            outcomes.add('L')
            outcomes.add('N')
            return outcomes
    # didn't find a winning move
    outcomes.add('R')
    outcomes.add('P')
    return outcomes
```

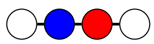
If we include a similar function for the right outcomes, then we can put them both together:

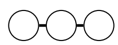
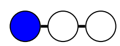
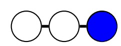
```
def get_outcome(left_outcomes, right_outcomes):
    outcomes_by_left = get_outcomes_by_left(left_outcomes)
    outcomes_by_right = get_outcomes_by_right(right_outcomes)
    outcomes = outcomes_by_left.intersection(outcomes_by_right)
    outcome = outcomes.pop()
    return outcome
```

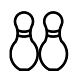
In that code, each of the inner functions returns a set of two possible outcome classes. We intersect those two sets to get the overall whole outcome class, then use the `pop` method to get what should be the only element in there.

Exercises for 2.2

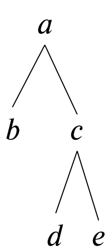
⋆ **0)** Draw out the entire game tree for  and label with outcome classes to show that it is in $\mathcal{L}$. Make sure your edges are pointing in the right direction! (Answer 2.2.0 in Appendix)

1) Draw out the entire game tree for  and label with outcome classes to show that it is in $\mathcal{P}$. Make sure your edges are pointing in the right direction!

2) Draw a trimmed game tree for  and find its outcome class. In order to keep the size reasonable, you can safely assume that outcome classes are equivalent if we rotate the board 180-degrees. E.g. $o($  $) = o($ ).

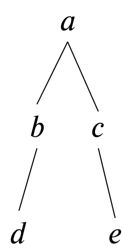
⋆ **3)** Draw out the entire (partisan) game tree for  . Label your tree with the outcome classes. Do you get the correct outcome class that you would using an impartial game tree? (Answer 2.2.3 in Appendix)

⋆ **4)** Label all positions of the following game tree with their outcome classes.

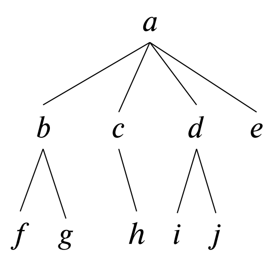


(Answer 2.2.4 in Appendix)

5) Label all positions of the following game tree with their outcome classes.

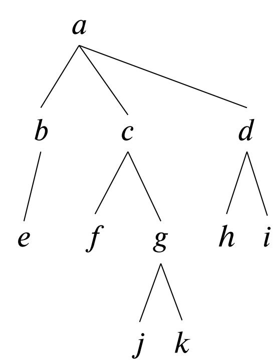


⋆ **6)** Label all positions of the following game tree with their outcome classes.

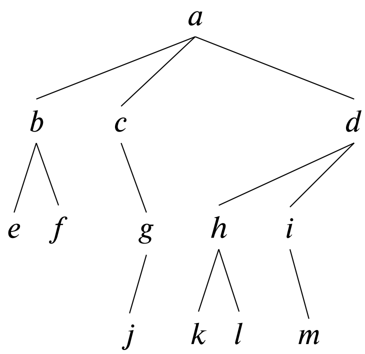


(Answer 2.2.6 in Appendix)

7) Label all positions of the following game tree with their outcome classes.

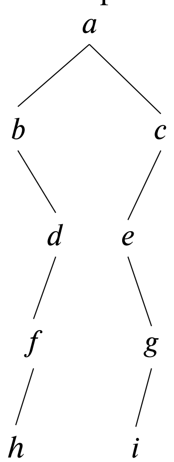


⋆ **8)** Label all positions of the following game tree with their outcome classes.



(Answer 2.2.8 in Appendix)

9) Label all positions of the following game tree with their outcome classes.



⋆ **10)** What is the smallest game tree you can draw that has an outcome class of $\mathcal{R}$? Can you find a COL position that has that tree? (Answer 2.2.10 in Appendix)

⋆ **11)** Translate this position into a game tree and determine the outcome class:

$$\left\{ \{\,0\,|\,\},\{\,|\,0\,\} \,\middle|\, \left\{ \{\,|\,0\,\} \,\middle|\, \{\,|\,0\,\} \right\} \right\}.$$ (Answer 2.2.11 in Appendix)

12) Translate this position into a game tree and determine the outcome class:

$$\left\{ \{\,|\,0\,\},\left\{ \,\middle|\, \{\,0\,|\,\} \right\} \,\middle|\, 0,\{\,0\,|\,0\,\} \right\}.$$

13) Translate this position into a game tree and determine the outcome class:
$\left\{ \left\{ \{0 \mid 0\} \,\middle|\, \right\}, \{0 \mid \} \,\middle|\, \{0 \mid \}, \left\{ \{0 \mid \} \,\middle|\, \right\} \right\}$.

You can simplify the game tree by not repeating analysis for positions after you find the outcome class once (in this problem).

14) Write `get_outcomes_by_right(right_outcomes)` to complete the example in the section.

2.3. Partisan Game Sums

Just as with impartial games, we can add partisan games together. The basic idea is the same as we saw in Section 0.4: if we add G and H together, then on any turn, the current player picks one of the two games to make a move on and does nothing to the other. The complication here, of course, is that the players have different sets of options on both G and H.

To help us define this formally, we'll first introduce some new notation. For any game, G, G^L ("the left options of G) is the set of options of G for L. Similarly, G^R ("the right options of G") is the set of options for R. For example, if $G = \{ 0, * \mid \}$, then $G^L = \{0, *\}$ and $G^R = \emptyset$.

This means that

$$G = \left\{ G^L \,\middle|\, G^R \right\}$$

even though we don't write out the extra squiggly-braces inside the formal game notation[5].

Now what happens if we add G and H? What are $(G + H)^L$ and $(G + H)^R$?

For L, they can either make on of their moves on G and add that to all of H, or make a move on H and add it to all of G. So:

$$(G + H)^L = \{g_l + H \mid g_l \in G^L\} \cup \{G + h_l \mid h_l \in H^L\}$$

Then, similarly,

$$(G + H)^R = \{g_r + H \mid g_r \in G^R\} \cup \{G + h_r \mid h_r \in H^R\}$$

[5]This is a bit of an abuse of notation, but it is commonly used in combinatorial game theory.

This means that

$$G+H = \left\{ \{g_l + H \mid g_l \in G^L\} \cup \{G + h_l \mid h_l \in H^L\} \,\middle|\, \{g_r + H \mid g_r \in G^R\} \cup \{G + h_r \mid h_r \in H^R\} \right\}$$

As you can imagine, this gets a bit onerous. We can simplify this definition, but we need to extend some other notation.

We can generalize our sum notation to work for single games added to sets of games so that if we add a single position, G, to a set of positions, $\mathcal{H}$, the sum is a new set of positions with G plus each element of $\mathcal{H}$:

$$G + \mathcal{H} = \mathcal{H} + G = \{G + H \mid H \in \mathcal{H}\}$$

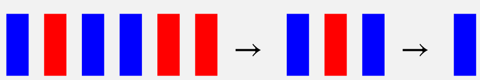
Now, with this new notation, we can rewrite our sum of G and H:

$$G + H = \left\{ (G^L + H) \cup (G + H^L) \,\middle|\, (G^R + H) \cup (G + H^R) \right\}$$[6]

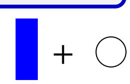
Let's include another partisan ruleset to our repetoire and then try out some game sums.

ToppLingDominoes

ToppingDominoes is a game played on rows of dominoes colored Red, Blue, or Green. Each turn, the current player picks either a green domino or domino of their color and a direction either left or right. (Either player can choose any direction.) The chosen domino and all other dominoes in the chosen direction are then "knocked down" and removed from play.



L makes the first move, knocking the fourth domino to the right. R makes the second move, knocking the second domino to the left.

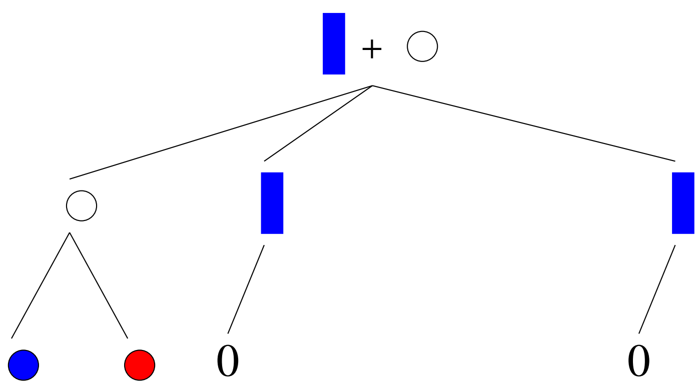
Let's do a simple sum example of Toppling Dominoes and Col: 
.

[6]You might see this instead written as $G + H = \left\{ G^L + H, G + H^L \,\middle|\, G^R + H, G + H^R \right\}$.

▮ $= \{ 0 \mid \}$ and ○ $= \{$ 🔵 $\mid$ 🔴 $\}$. Thus, ▮ + ○ $= \{ 0 +$ ○, ▮ + 🔵 $\mid$ ▮ + 🔴 $\}$.
Let's quickly cover each of those options in the sum.

- $0 +$ ○ $=$ ○ is the result if L knocks over the domino on their turn. (This is an element of the form $g_l + H$.)
- ▮ + 🔵 = ▮ is the result if L fills in the COL node. (This is an element of the form $G + h_l$.)
- ▮ + 🔴 = ▮ is the only option for R; they play on the COL component. (This is an element of the form $G + h_r$.)
- There is no element of the form $g_r + H$ because R doesn't have any moves in the TOPPLING DOMINOES component.

The sum game tree (with some of the elements simplified) looks like this:



By labelling the outcome classes, we can see that the overall outcome class is $\mathcal{L}$. The individual components, ▮ and ○ are in $\mathcal{L}$ and $\mathcal{N}$ respectively. (More specifically, we know one of the values, as ○ $= \{ 0 \mid 0 \} = {}^{*}\{ 0 \} = *$. We don't (yet) have a symbol for the value of ▮, though we will in section 3.1.)

It would be nice and easy if it turned out that the sum of two games, one in $\mathcal{L}$ and one in $\mathcal{N}$, always resulted in a game in $\mathcal{L}$, but this is not the case. Consider $\{\, 0 \mid * \,\} + \{\, 0 \mid 0 \,\}$:

- $\{\, 0 \mid * \,\} \in \mathcal{L}$, because L has a winning move, but R doesn't.
- $\{\, 0 \mid 0 \,\} = * \in \mathcal{N}$, as we already know.
- But, $\{\, 0 \mid * \,\} + \{\, 0 \mid 0 \,\} \in \mathcal{N}$, because L has a winning move (to $\{\, 0 \mid * \,\}$) and R also has a winning move (to $* + * = 0$).

Thankfully, it does turn out that these are the only two options. In general, we can make a table to describe the possible outcome classes of sums:

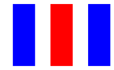
$+$	$\mathcal{L}$	$\mathcal{P}$	$\mathcal{N}$	$\mathcal{R}$
$\mathcal{L}$	$\mathcal{L}$	$\mathcal{L}$	$\mathcal{L}$ or $\mathcal{N}$	anything
$\mathcal{P}$	$\mathcal{L}$	$\mathcal{P}$	$\mathcal{N}$	$\mathcal{R}$
$\mathcal{N}$	$\mathcal{L}$ or $\mathcal{N}$	$\mathcal{N}$	anything	$\mathcal{N}$ or $\mathcal{R}$
$\mathcal{R}$	anything	$\mathcal{R}$	$\mathcal{N}$ or $\mathcal{R}$	$\mathcal{R}$

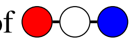
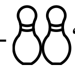
Some things to notice about these outcome sums:

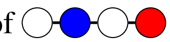
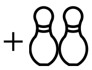
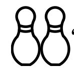
- Components in $\mathcal{P}$ don't change the overall outcome class at all. This works out well, because $\mathcal{P}$ continues to act like the value 0.
- $\mathcal{L} + \mathcal{L} \to \mathcal{L}$ and $\mathcal{R} + \mathcal{R} \to \mathcal{R}$, as we might expect.
- $\mathcal{N}$ makes things pretty complicated since it includes more values than just the non-zero nimbers, such as $\{\, 0 \mid * \,\} + *$, as we saw before.

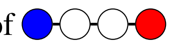
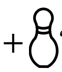
When we start learning about more game values, we'll be able to use sums more fully, just as we did with impartial games. In the meantime, many of the combinations do result in one outcome class. If it doesn't, we'll have to use the sum's game tree to reason about the overall class.

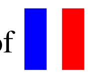
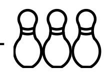
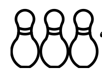
Exercises for 2.3

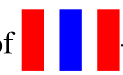
0) Consider the following TOPPLING DOMINOES position: 
Use a game tree to determine the outcome class. Justify your answer. Hint: you can save some time by reusing positions, but make sure all the arrows are pointing in the correct directions.

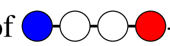
⋆ **1)** What is the outcome class of  + ? (Answer 2.3.1 in Appendix)

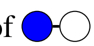
2) What is the outcome class of  +  ?

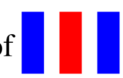
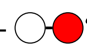
3) What is the outcome class of  + ?

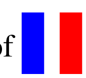
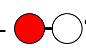
⋆ **4)** What is the outcome class of  +  ? (Answer 2.3.4 in Appendix)

5) What is the outcome class of  +  ?

6) What is the outcome class of  +  ?

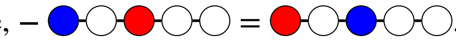
⋆ **7)** What is the outcome class of  + ? (Answer 2.3.7 in Appendix)

8) What is the outcome class of  + ?

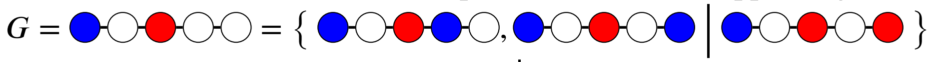
⋆ **9)** What is the outcome class of  + ? (Answer 2.3.9 in Appendix)

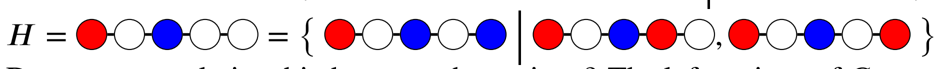
2.4. Negatives and Equality

So far we know how to add positions together. It would be helpful to be able to subtract as well. In normal math, $x - y$ is the same as $x + (-y)$. The same is true for combinatorial games; $G - H = G + (-H)$. That means that in order to subtract games, we can use our game addition if we know how to negate a position.

For many rulesets, negating a position is easy in practice (with actual game pieces) and more difficult notationally. Often you can get the negative just by swapping all the pieces. For example, .

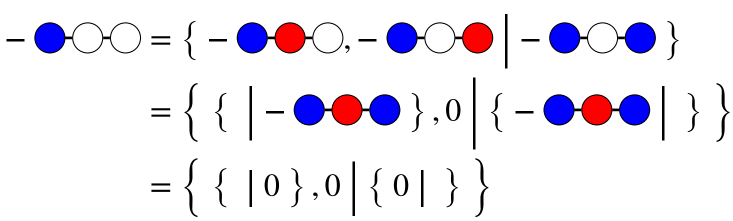
Let's take a look and see what the options for these two opposite games are:



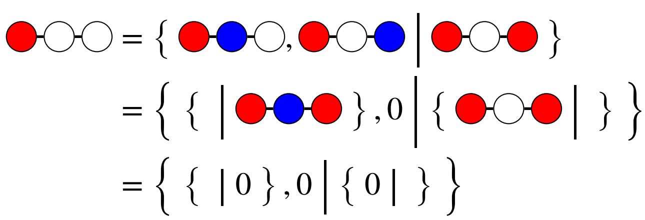


Do you see a relationship between the options? The left options of G are negatives of the right options of H and the right options of G are negatives of the left options of H. In fact, this is exactly how we define the negation operation: swap the sides the options are on and negate them each. Using our notation, if $G = \left\{ g_1^L, g_2^L, g_3^L, \ldots, g_n^L \;\middle|\; g_1^R, g_2^R, \ldots, g_m^R \right\}$, then $-G = \left\{ -g_1^R, -g_2^R, \ldots, g_m^R \;\middle|\; -g_1^L, -g_2^L, \ldots, -g_n^L \right\}$.

Let's look at a simpler example, and go all the way down to the zeroes to make sure that the color-swapping plan holds up:



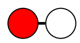
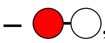
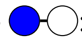
and

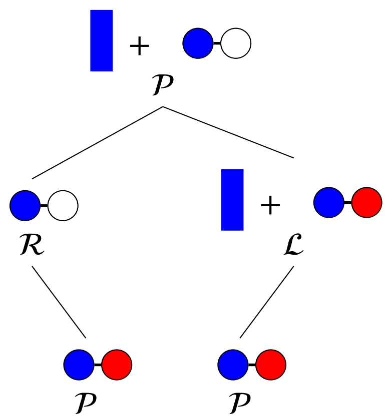


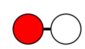
Indeed, swapping the colors has the same effect as negating a COL position!

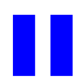
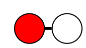
Negation is also a useful tool for showing that two positions are equivalent. Just as with numbers, two games are equivalent if their difference is zero:

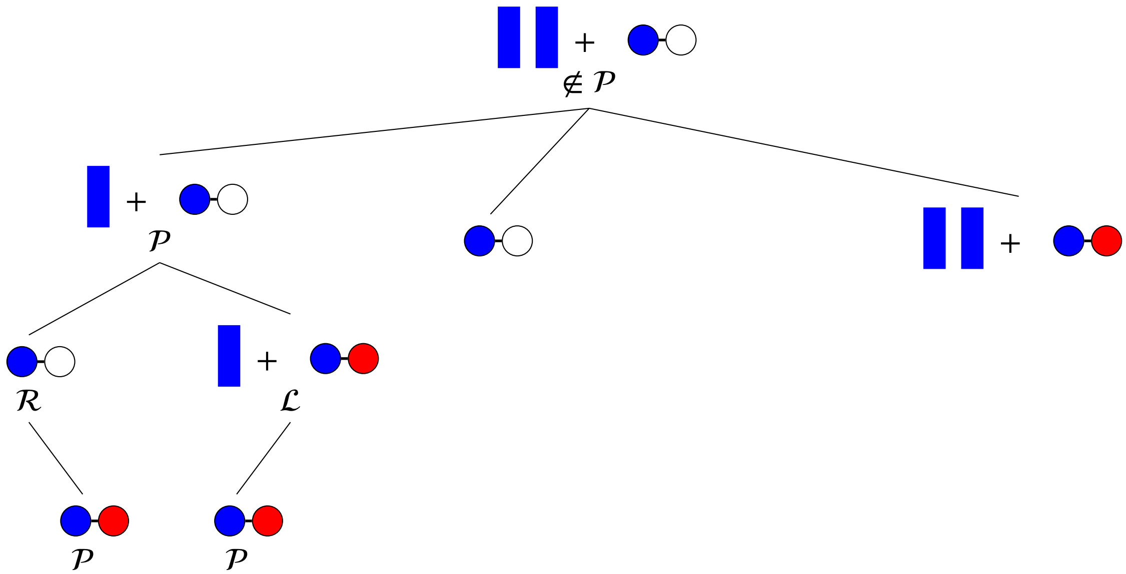
$$G = H \Leftrightarrow G - H = 0 \, (\Leftrightarrow G - H \in \mathcal{P})$$

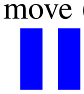
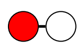
For example, we might want to determine whether  is equal to the Col position . We can do this by finding the outcome class of  $-$ , which is equal to  $+$ :



Since the outcome class of the difference is $\mathcal{P}$,  $=$ .

As another example: what happens if we check whether  equals ?



L has a winning move (to the zero we found before) so the sum cannot be in $\mathcal{P}$. This means that  $\neq$ .

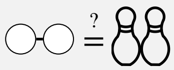
It's important to point out that "equals" here doesn't necessarily mean "exactly the same as". The two positions $\{\ |\ \}$ and $\left\{ \{\ |0\} \,\middle|\, \{0|\ \} \right\}$ are both equal to 0, even though the second one has (losing) options for both players.

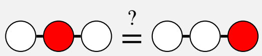
It will also be helpful to know that any impartial game is always its own negative: $-*n = *n$. Unfortunately, there are some elements of $\mathcal{N}$ that are not strictly impartial, and are thus not equal to their own negative. ($\mathcal{P}$ does not have this problem, because $-0 = 0$.) In general,

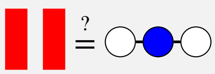
$o(G)$	$\mathcal{L}$	$\mathcal{P}$	$\mathcal{N}$	$\mathcal{R}$
$o(-G)$	$\mathcal{R}$	$\mathcal{P}$	$\mathcal{N}$	$\mathcal{L}$

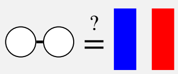
Activity: Testing Equality

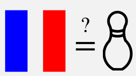
Sit down with a friend and use negatives to play some games to figure out whether the equality holds between these games or not: (Hints: TOPPLING DOMINOES positions can be negated by flipping the colors of the dominoes.)

1. 

2. 

3. 

4. 

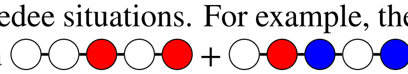
5. 

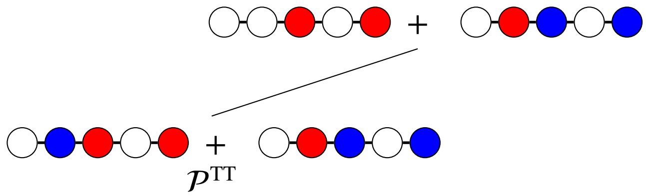
We now have a tool to determine whether any two game positions are equivalent. As we continue, we will learn tricks to expedite this process. Here are a few we already know that we can use in specific cases:

- If they are both impartial, then we can find the nimbers to show either $G = H$ or $G \neq H$ (depending on whether the nimbers are equal). If one (or both) isn't impartial, but is still equal to a nimber, then we can still use those nimbers.

- If $G = 0$, then we can just check whether H is also equal to zero.

- If they are in different outcome classes, we can show this to prove $G \neq H$. Unfortunately, if they are in the same outcome class (and that class isn't $\mathcal{P}$) this isn't enough to show that they are equal.

When playing games, it is extremely helpful to consider negatives to set up Tweedledum-Tweedledee situations. For example, there is a clear winning move for L playing next on .



This move isn't even L's best move[7], but in this case it doesn't matter because it's an obvious win.

Notice that we used the Tweedledum-Tweedledee-notation (TT) there, even though it doesn't follow the previous rule of $G + G$ for impartial games. This is the new (and improved) version of Tweedledum-Tweedledee that you can use, anytime it is obvious that the position-in-question is equal to $G - G$ for some G.[8] This usually means that it's clear that you have a sum of a position and it's opposite. If it's not obvious that the second component is the negation of the first, but we've already determined that they are negatives, then that determination is better than Tweedledum-Tweedledee (assuming it's correct and the work is shown).

We have cheated a little bit here, and explained how to determine equivalence without giving a clear definition. Formally, $G = H$ if:

$$\forall \text{ positions } X : o(G + X) = o(H + X)$$

In (other) words, G equals H if, when you add G and H each to any other game, those sums are always in the same outcome class.

Even though this is the definition of equality, in practice we do what we've been doing so far in this section. We use the method of finding the outcome class of the difference to actually determine (and prove) equality between two games.

Let's quickly restate all of the tricks we have now to determine the equality of two games, say G and H. (We'll need these tricks for the exercises.)

- If $G = H$, we can prove that by:
  - Proving that they are both in $\mathcal{P}$, or

[7] Can you find their best move?

[8] The previous $G + G$ still works for impartial positions, as $-G = G$, so $G - G = G + G$.

  - Proving they are both equal to the same nimber, $*k$, or
  - Proving they have the same notation, or
  - Proving that $G - H = 0$.

- If $G \neq H$, we can prove that by:
  - Proving that they are in different outcome classes, or
  - Proving that they are equal to different nimbers, or
  - Proving that one is equal to a nimber and the other isn't, or
  - Proving that $G - H \neq 0$.

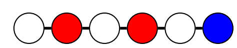
We will learn more ways to do this in future sections. It's also an important reminder that there are things that may seem like working proofs that are not:

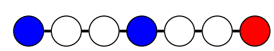
- We cannot prove that $G = H$ by proving that they are in the same outcome class (unless that class is $\mathcal{P}$).

- We cannot prove that $G \neq H$ by proving that they have different notation. We will see that positions with different notation can still have the same value. (We will see many tricks about how to simplify notation to often solve this problem.)

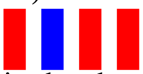
If you're not sure whether two positions, G and H are equal, it's recommended to check their outcome classes first. If they are the same, then check the difference $G - H$.

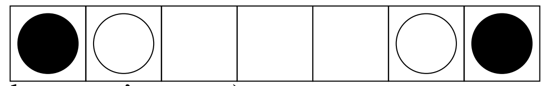
Exercises for 2.4

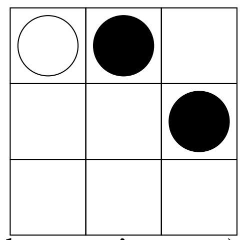
⋆ **0)** Find the COL position that is the (obvious) negation of . (You don't need to show work if you use the (super fast) trick given in the chapter.) (Answer 2.4.0 in Appendix)

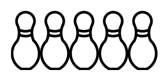
1) Find the COL position that is the (obvious) negation of . (You don't need to show work if you use the (super fast) trick given in the chapter.)

⋆ **2)** Find the TOPPLING DOMINOES position that is the (obvious) negation of . (You don't need to show work if you use the (super fast) trick given in the chapter.) (Answer 2.4.2 in Appendix)

3) Find the TOPPLING DOMINOES position that is the (obvious) negation of . (You don't need to show work if you use the (super fast) trick given in the chapter.)

⋆ **4)** Find the KONANE position that is the (obvious) negation of . (Hint: KONANE positions can be negated just like the other rulesets we've seen.) (Answer 2.4.4 in Appendix)

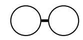
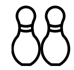
5) Find the KONANE position that is the (obvious) negation of . (Hint: KONANE positions can be negated just like the other rulesets we've seen.)

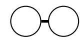
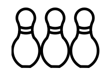
⋆ **6)** Find the KAYLES position that is the (obvious) negation of . (Hint: what is the negation of a position from an impartial ruleset?) (Answer 2.4.6 in Appendix)

7) Find the NIM position that is the (obvious) negation of . (Hint: what is the negation of a position from an impartial ruleset?)

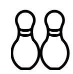
⋆ **8)** Simplify $G = -\{\, 0, *2 \,|\, 0, *2 \,\}$ by removing the minus sign and reducing your answer as much as you know how. (Answer 2.4.8 in Appendix)

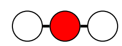
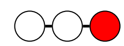
9) Simplify $G = -\left\{\, 0, *, *2 \,\middle|\, \{\ |\ \}, \{\, 0 \,|\, 0 \,\} \,\right\}$ by removing the minus sign and simplifying your answer as much as you know how.

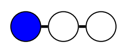
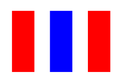
⋆ **10)** Is COL position  equal to  ? Prove your answer. (Yes, this is from one of the team exercises.) (Answer 2.4.10 in Appendix)

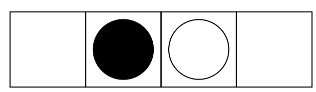
⋆ **11)** Is COL position  equal to  ? Prove your answer. (Answer 2.4.11 in Appendix)

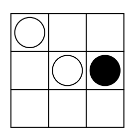
⋆ **12)** Is COL position  equal to  ? Prove your answer. (Answer 2.4.12 in Appendix)

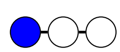
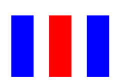
⋆ **13)** Is COL position  equal to  ? Prove your answer. (Answer 2.4.13 in Appendix)

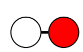
⋆ **14)** From another of the team exercises, is  equal to  ? (Both are COL.) Prove your answer. (Answer 2.4.14 in Appendix)

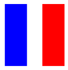
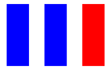
⋆ **15)** Is COL position  equal to  ? Prove your answer. (Answer 2.4.15 in Appendix)

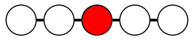
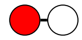
16) Is KONANE position  equal to  ? Prove your answer.

17) Is KONANE position  equal to   ? Prove your answer.

18) Is COL position  equal to  ? Prove your answer.

19) Is  equal to COL position  ? Prove your answer.

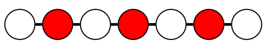
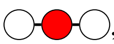
20) Is  equal to ? Prove your answer.

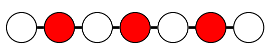
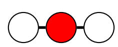
21) Is COL position  equal to ? Prove your answer. (Gather your papers, the solution to this one is quite long.)

2.5. Inequalities

Checking whether two game positions are equivalent is not the only way to compare them. Perhaps we can also use inequalities similar to those used with numbers. Since we have given two of our outcome classes the names “positive” and “negative”, we might want this:

- $G > H$ means “G is better for L than H”
- $G < H$ means “G is better for R than H”

This does not mean that the outcome classes, $o(G)$ and $o(H)$, have to be different. For example,  can be considered better for L than , even though they are both in $\mathcal{L}$. In the first, L has four separate plays they can make, while the second has only two for L. The first position is better for L, so we want to be able to say that it’s “greater than” the second.

 > 

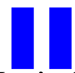
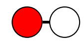
How can we formalize this notion? How can we check whether this relationship holds for any two positions? Well, just as we saw in the last section, the way to actually compare is to find the outcome class of the difference of the two games. As you might (or might not) be expecting, there is an extra wrinkle here. Here are the four possibilities we get from analyzing the outcome class of a difference of game positions:

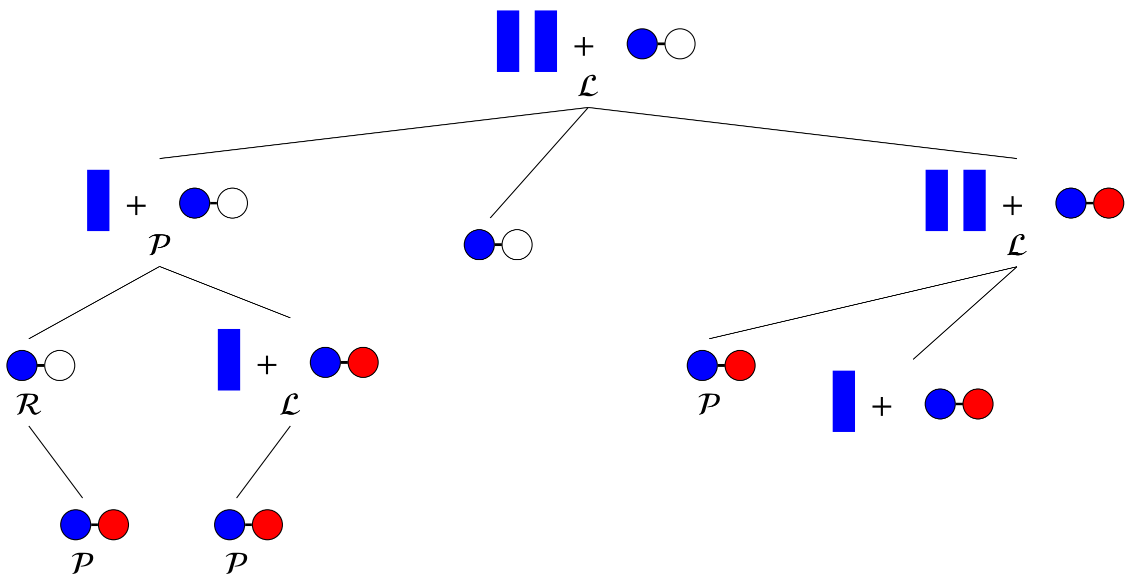
- If $o(G - H) = \mathcal{P}$, then $G = H$.
- If $o(G - H) = \mathcal{L}$, then $G > H$.
- If $o(G - H) = \mathcal{R}$, then $G < H$.

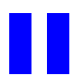
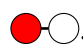
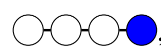
- If $o(G - H) = \mathcal{N}$, then $G \parallel H$.

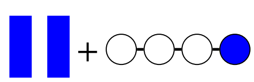
This last case, $G \parallel H$ means "G is confused with H" or "G is incomparable to H". For example, $*$ is confused with $*2$. They are not greater than, nor less than, nor equal to each other.

In the previous section, we saw examples of equivalent positions. Let's see some examples of the other three.

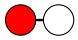
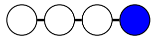
First, what is the relationship between  and  ? This is one of the examples we started in the last section. Let's finish the analysis by finding the outcome class of the difference:

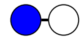
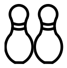


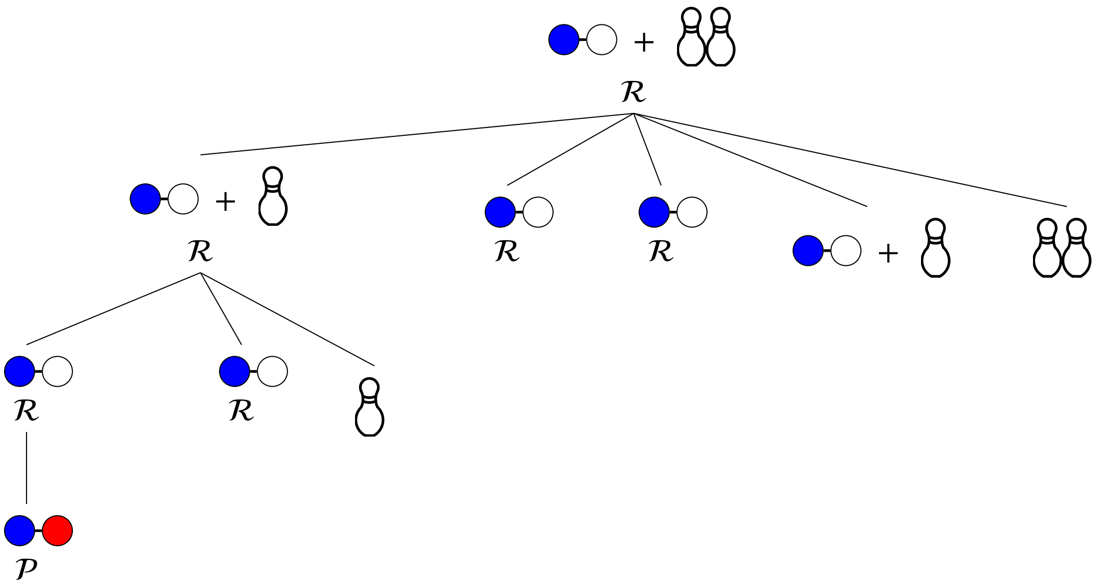
Since the difference is in $\mathcal{L}$,  > . To reinforce that this matches our definition, we can consider a separating game, X = , to add to both sides. You can check that both:

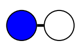
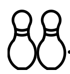
 $\in \mathcal{L}$

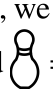
and

 +  $\in \mathcal{R}$

As another example, let's determine the relationship between  and . Just as before, we look at the outcome class of the difference.[9]



This difference is in $\mathcal{R}$, so  < .

In our last example, let's find the relationship between  and . Since these are both impartial, we can analyze this without using a game tree! We know that  $= *3$ and  $= *$, so their difference is equal to their sum, which is $*3 + * = *2$.

Since the difference is in $\mathcal{N}$, we know that  $\parallel$ .

As with numbers, we can combine or negate some symbols to get additional (binary) relations (e.g. $\leq$). Here's what we have with games:

[9] Recall that impartial game positions are their own negatives (section 2.4).

- $G \leq H$ ("G is less than or equal to H") means $G < H$ or $G = H$. (*geq* is analagous)
- $G \not\parallel H$ ("G is not confused with H") means either $G > H$, $G < H$, or $G = H$.
- $G \lhd\!\!\parallel H$ ("G is less than or confused with H") means either $G < H$ or $G \parallel H$.
- $G \parallel\!\!\rhd H$ ("G is greater than or confused with H") means either $G > H$ or $G \parallel H$.
- $G \parallel\!\!\not\rhd H$ ("G is not greater than or confused with H") means neither $G > H$ nor $G \parallel H$. This is equivalent to $G \leq H$.
- $G \not\lhd\!\!\parallel H$ ("G is not less than or confused with H") means neither $G < H$ nor $G \parallel H$. This is equivalent to $G \geq H$.

Sometimes we can determine the relationship without using the difference. If we know the two outcome classes, we might be able to deduce the difference, or at least eliminate some possibilities. Here's a table showing the different possible relationships of G and H, based on their outcome classes:

		H: $\mathcal{L}$	H: $\mathcal{P}$	H: $\mathcal{N}$	H: $\mathcal{R}$
G	$\mathcal{L}$	anything	$G > H$	$G > H$ or $G \parallel H$	$G > H$
	$\mathcal{P}$	$G < H$	$G = H$	$G \parallel H$	$G > H$
	$\mathcal{N}$	$G < H$ or $G \parallel H$	$G \parallel H$	anything	$G > H$ or $G \parallel H$
	$\mathcal{R}$	$G < H$	$G < H$	$G < H$ or $G \parallel H$	anything

An important note about $\mathcal{N}$: so far we have seen the nimbers ($*, *2, *3, \ldots$) but there exist many other values in $\mathcal{N}$. In the table above, the $\mathcal{N}$-$\mathcal{N}$ cell signifies that the difference can be in any outcome class. However, from the small subset of $\mathcal{N}$ values we've seen (the nimbers) we can't get a difference in $\mathcal{L}$ or $\mathcal{R}$. Whenever you subtract (or add) two nimbers, the value is always another nimber or zero.[10]

Let's use these inequalities to prove something about options.

[10] Yes, you can use this fact in the exercises below.

Theorem 2.5.1 (Option Constraints). *If H is a left-option of G, then $H \lhd\!\| \, G$.*

Proof. Proof-by-contradiction. Assume that $H \geq G$. Thus, $H - G \geq 0$, so $H - G \in \mathcal{L} \cup \mathcal{P}$. Since H is a left-option of G, $-H$ is a right-option of $-G$. Thus, $H - H = 0$ is a right-option of $H - G$. Thus, R has a winning move on $H - G$, so $H - G \in \mathcal{R} \cup \mathcal{N}$. $\rightarrow\!\leftarrow$.

Thus, it cannot be that $H \geq G$, meaning $H \lhd\!\| \, G$. □

Corollary 2.5.2. *If H is a left-option of G and both are numbers, then $G > H$.*

Proof. Direct proof. Since G and H are both numbers, $G = H$ is also a number. Since $G \,\|\!\rhd H$, $G - H$ cannot be negative nor zero. Thus, $G - H > 0$. □

Corollary 2.5.3. *If H is a right-option of G, then $H \,\|\!\rhd G$.*

We leave the proof for exercise 2.5.15 .

Now that we have a sense for comparing games (or not, as in the case of incomparable games), let's take some space to address the mathematics behind these kinds of relationships.

Math Diversion: Relations

A mathematical *relation* is a way of comparing two sets. Formally, a relation R is a subset of the cartesian product of sets A and B, i.e.

$$R \subseteq A \times B$$

so that, if $a \in A, b \in B$, and $(a, b) \in R$ then we say that *a is related to b by the relation R*, or even that aRb. When R is known from context, we sometimes write this as $a \sim b$.

Relations should remind you of functions. In fact, every function is a relation but not every relation is a function, since R can contain both (a, b_1) and (a, b_1).

A relation can have any of the following properties:

1. R is *reflexive* if xRx for every x.

2. R is *symmetric* if xRy whenever yRx/

3. R is *transitive* if whenever xRy and yRz, we also have xRz.

Some relations that are not symmetric are *antisymmetric*. This means that if xRy and $x \neq y$, that it cannot ever be the case that yRx.

For example, the relation '$>$' on the set $\mathbb{R}$ is transitive, but neither reflexive nor symmetric. This is because, while $x > y, y > z$ implies that $x > z$, we do not have that $x > x$ nor that $x > y$ implies that $y > x$ for all real x and y.

Which properties does the relation '$\leq$' have on the set $\mathbb{Z}$?

A: *It is reflexive since $x \leq x$ for all $x \in \mathbb{Z}$, and transitive since $x \leq y, y \leq z \Rightarrow x \leq z$. But it is not symmetric since $1 \leq 5$ but it is not true that $5 \leq 1$. In fact, $\leq$ is antisymmetric.*

An *equivalence relation* is a relation that has all three of these properties. A simple example of an equivalence relation on the set of all people is "has the same first name as". Note that everyone has the same first name as themself, if person x has the same first name as person y then person y has the same first name as person x. You should convince yourself that this relation is transitive.

An equivalence relation also provides a nice way to partition a set.

It's relatively straightforward to show that "=" is an equivalence relation on any set of numbers. What about game equality? Recall that games G and H are called *equal* if $o(G + X) = o(H + X)$ for any game X. Let's prove directly that game equality is an equivalence relation.

Claim 2.5.4. *The relation "=" on games is an equivalence relation.*

Proof. We must show that this relation is reflexive, symmetric, and transitive.

- For any game G and any game X, $o(G + X) = o(G + X)$ since the games are identical, so "=" is reflexive.

- Say that we have games G and H such that $o(G + X) = o(H + X)$ for any game X. Then it is clear that we also have $o(H + X) = o(G + X)$ for any game X, hence $G = H \Rightarrow H = G$ and "=" is symmetric.

- If $G = H$ and $H = J$ for games G, H, and J, then for any games X and Y, $o(G + X) = o(H + X)$ and $o(H + Y) = o(J + Y)$. If we let Z be any game, then $o(G + Z) = o(H + Z)$ and $o(H + Z) = o(J + Z)$, so $o(G + Z) = o(J + Z)$. Hence game equality is also transitive, and is therefore an equivalence relation.

□

Now what do we make of relations like $<, \leq, |\triangleright$ etc.? We need to talk a bit about orders.

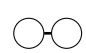
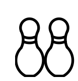
On $\mathbb{R}$ or any of its subsets, the relations $<, \leq, >, \geq$ are reflexive, transitive, and antisymmetric. This makes them into *partial orders*. A set with a partial order is called a *partially ordered set* or a*poset*. Another nice example of a partial order is $\subseteq$ on the set of sets.

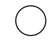
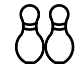
Claim 2.5.5. *The relation $\subseteq$ on the set of sets is a partial order.*

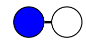
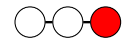
Proof. Let A, B, and C be any three sets. $A \subseteq A$ since $A = A$, so $\subseteq$ is reflexive. It is antisymmetric since the only time we have both $A \subseteq B$ and $B \subseteq A$ is when $A = B$. To show transitivity, let $a \in A$. If $A \subseteq B$ then $a \in B$, and if $B \subset C$ then every element in B is also in C. Hence, $a \in C$ for all $a \in A$. So $\subseteq$ is a partial order. □

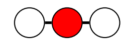
A *total order* is a partial order with the additional property that *every* pair of elements is comparable. That is, if x, y are in the set then xRy or yRx. Note that we do not have a total order on the set of all games since some games are incomparable. We will investigae the relations $\lhd|$ and $||$ once we have some more tools at our disposal.

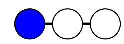
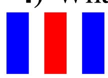
Exercises for 2.5

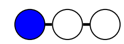
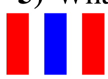
⋆ **0)** What is the inequality relationship between COL position  and  ? Prove your answer. (This is a follow-up of exercise 2.4.10.) (Answer 2.5.0 in Appendix)

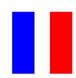
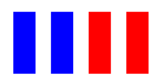
★ **1)** What is the inequality relationship between COL position  and  ? Prove your answer. (This is a follow-up of exercise 2.4.13.) (Answer 2.5.1 in Appendix)

★ **2)** What is the inequality relationship between COL positions  and  ? Prove your answer. (Answer 2.5.2 in Appendix)

3) What is the inequality relationship between COL positions  and ? Prove your answer. (This is a follow-up of exercise 2.4.14.)

4) What is the inequality relationship between COL position  and  ? Prove your answer. (This is a follow-up of exercise 2.4.18.)

5) What is the inequality relationship between COL position  and  ? Prove your answer. (This is a follow-up of exercise 2.4.15.)

6) What is the inequality relationship between  and  ? Prove your answer.)

★ **7)** $G \in \mathcal{L} \cup \mathcal{N}$. Rewrite this as an equivalent comparison of G and 0. (Answer 2.5.7 in Appendix)

8) $G \in \mathcal{L} \cup \mathcal{P}$. Rewrite this as an equivalent comparison of G and 0.

9) $G \in \mathcal{P} \cup \mathcal{N} \cup \mathcal{R}$. Rewrite this as an equivalent comparison of G and 0.

★ **10)** $G \not\gtrsim 0$. Rewrite this as an equivalent comparison of G and 0 using a different symbol. (Answer 2.5.10 in Appendix)

11) $G \not\mathrel{|\!\triangleright} 0$. Rewrite this as an equivalent comparison of G and 0 using a different symbol.

★ **12)** $G \leq 0$. Write the equivalent expression of G as an element of the union of outcome classes. (Answer 2.5.12 in Appendix)

13) $G \neq 0$. Write the equivalent expression of G as an element of the union of outcome classes.

14) $G \mathrel{\lhd\!\!\|} 0$. Write the equivalent expression of G as an element of the union of outcome classes.

15) Prove the Corollary in the section: If H is a right-option of G, then $H \mathrel{\|\!\!\rhd} G$.

⋆ **16)** Is the relation "has the same decimal part as" on the set of all reals reflexive, symmetric, and/or transitive? Demonstrate. (Answer 2.5.16 in Appendix)

17) Is the relation "is taller than" on the set of all people in your class reflexive, symmetric, and/or transitive? Demonstrate.

⋆ **18)** Is the relation "has the same remainder when divided by 3" on the set of all naturals reflexive, symmetric, and/or transitive? Demonstrate. (Answer 2.5.18 in Appendix)

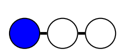
19) Is the relation "is smaller than the square of" on the set of all reals reflexive, symmetric, and/or transitive? Demonstrate.

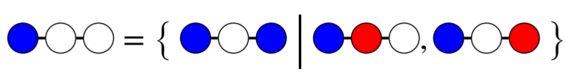
20) Is the relation "is within 100 miles of" on the set of all towns reflexive, symmetric, and/or transitive? Demonstrate.

2.6. Dominated Options

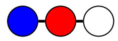
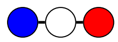
Now that we know how to compare games, we can use this information to simplify some positions. When a player is choosing from amongst the options of a position, they might want to pick something different depending on what else the position is added to (if anything). Sometimes it's better to choose $*$; sometimes it's better to choose 0.

However, we can eliminate some options from consideration if they're always worse than the others.

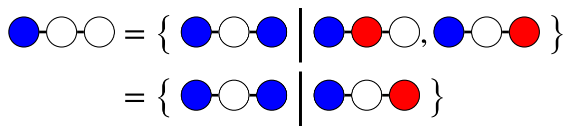
For example, consider this Col position:  :



Let's consider those options for R:

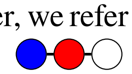
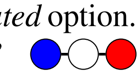
-  has a move for L (to zero) and no moves for R, so it's in $\mathcal{L}$.
-  has no moves for either player, so it's in $\mathcal{P}$.

R will never prefer giving L a move over moving to 0, so they will never choose that option. That means we can remove it from evaluation of the position. Thus:

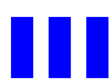


How do we know when a player will never prefer one option, G, over another, H? This happens exactly when the positions are neither equal nor confused with each other. If there is a strict greater-or-less-than relationship, then we can remove one of them. Here are all four cases:

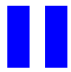
- $G > H$: If these are options for L, then we can remove H, as L will always prefer G. If these are options for R, then we can similarly remove G.
- $G = H$: the positions will have the same value, so we can remove either one of them (but not both).
- $G \parallel H$: the positions are confused with each other. We can't remove either of them.
- $G < H$: If these are options for L, then we can remove G. Otherwise, we can remove H.

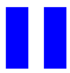
Whenever we drop an option for a player, we refer to that as a *dominated* option. In the example above, we would say that  "is dominated by"  "for R". Sometimes one option will dominate many others, which can greatly simplify your analysis.

In practice, dominating options happen all the time. It's easy to eliminate moves that are clearly bad for you. For example, in



L can choose to remove 1, 2, or 3 blue dominoes as their move. It is obvious to players, even those who haven't learned any combinatorial game theory, that it's best to only knock over one of the dominoes, leaving two for later. In the understanding we've built up, that's because

 $>$  > 0

and thus  dominates the other two options for L.

Exercises for 2.6

⋆ **0)** $G = \left\{ \{ \mid 0 \}, 0 \,\middle|\, 0 \right\}$. Determine whether any of G's options are dominated. If they are, remove them and simplify G as much as you know how. Show all your work. (Answer 2.6.0 in Appendix)

1) $G = \left\{ \,\middle|\, 0, \{ * \mid 0 \} \right\}$. Determine whether any of G's options are dominated. If they are, remove them and simplify G as much as you know how. Show all your work.

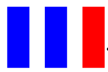
⋆ **2)** $G = \left\{ \{ 0 \mid \}, 0 \,\middle|\, \{ 0 \mid \}, 0 \right\}$. Determine whether any of G's options are dominated. If they are, remove them and simplify G as much as you know how. Show all your work. (Answer 2.6.2 in Appendix)

3) $G = \left\{ \{ 0 \mid \}, 0 \,\middle|\, \{ \mid 0 \}, 0 \right\}$. Determine whether any of G's options are dominated. If they are, remove them and simplify G as much as you know how. Show all your work.

⋆ **4)** $G = \{ *, 0 \mid 0, * \}$. Determine whether any of G's options are dominated. If they are, remove them and simplify G as much as you know how. Show all your work. (Answer 2.6.4 in Appendix)

5) $G = \left\{ *, 0 \,\middle|\, \{ 0 \mid * \} \right\}$. Determine whether any of G's options are dominated. If they are, remove them and simplify G as much as you know how. Show all your work.

6) $G = \left\{ 0, *2, \{ * \mid 0 \} \,\middle|\, \{ 0 \mid \; \}, 0, *3 \right\}$. Determine whether any of G's options are dominated. If they are, remove them and simplify G as much as you know how. Show all your work.

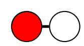
7) $G =$ . Which of the options are the dominating options for each player? (You do not need to do this formally.) Write G is as simplified set-notation as you know. What is $o(G)$?

3. Values

So far we only have shortcut values for zero and the nimbers. There are many more values for combinatorial game positions, though. Having these values will make it easier for us to make decisions while playing games.

3.1. Integers

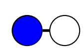
In this section, we will finally use the integers, $\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$, to label some game positions. We have already used 0 to label all positions in $\mathcal{P}$, but the rest of $\mathbb{Z}$ remains untouched.

We have seen the position $\{\, 0 \mid \,\}$ a few times.  and  are both equal to $\{\, 0 \mid \,\}$. This represents one extra move (to zero) for L, so let's use 1 for that.

$$1 = \{\, 0 \mid \,\}$$

We already know about negatives, so -1 should be

$$\begin{aligned} -1 &= -\{\, 0 \mid \,\} \\ &= \{\, \mid -0 \,\} \\ &= \{\, \mid 0 \,\} \end{aligned}$$

This makes sense, as this is one move only for R. Both  and  have this value.

We would like 2 and -2 to be similar: 2 should mean that L has two "free" moves; -2 should be worth two moves for only R. That means that if L makes a move on 2, the result should be 1. Thus:

$$2 = \{\, 1 \mid \,\}$$
$$= \{\, \{\, 0 \mid \,\} \mid \,\}$$

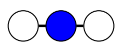
In the same way, 56 should have a single move to 55: $56 = \{\, 55 \mid \,\}$. With this in mind, we can define all of our positive integers. For any $n \in \mathbb{Z}_{>0}$:

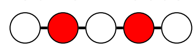
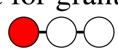
$$n = \{\, n - 1 \mid \,\}$$

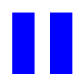
And, for the negatives:

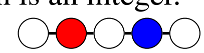
$$-n = \{\, \mid -(n-1) \,\}$$

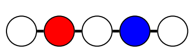
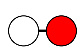
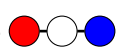
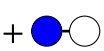
The most important part of these numbers is that they sum exactly as one would expect. For example, we can quickly say that

 $+$  $= -2 + 1 = -1$

Now that we have an understanding of integers, we no longer have to carefully analyze positions that are clearly a set number of moves for a single player. For example, we can take it for granted that  $= 3$. We can't do this for positions such as  , as there are some overlapping moves for the players. (Indeed, that value isn't even an integer.)

TOPPLING DOMINOES positions are similar.  is still 2, even though it has moves for L to both 1 and 0.[1] However, if there are both Red and Blue dominoes in the same row, then we can't just assume the position is an integer.

We can also quickly figure out the value of positions such as  . Since the uncolored vertices are all independent of each other, we can break it apart:

 $=$  $+$  $+$  $= 1 + 0 + -1 = 0$

Lots of things work out the way we want them to:

[1]In exercise 3.1.2, we ask the reader to prove this position equals 2.

- All positive integers are in $\mathcal{L}$.
- All negative integers are in $\mathcal{R}$.
- $\cdots 3 > 2 > 1 > 0 > -1 > -2 > -3 \cdots$

Unfortunately, when we add integers to nimbers, we can't simplify between the two. Thus, the terms of $3+*2$ cannot be combined, though $3+*2-1$ can be simplified to $2+*2$.

Alternatively, even though it doesn't simplify things, per se, we might want to rewrite a sum as a single game. Now that we know about integers, we can do that a little easier. For example, if we want to find the sum of

$$\{\, 2 \mid 1 \,\} + 3$$

we can use the definition of game sums to combine it into one position, then use arithmetic to simplify some of the options:

$$\begin{aligned}\{\, 2 \mid 1 \,\} + 3 &= \{\, 2 \mid 1 \,\} + \{\, 2 \mid \,\} \\ &= \left\{\, 2+3, \{\, 2 \mid 1 \,\} + 2 \,\middle|\, 1+3 \,\right\} \\ &= \left\{\, 5, \{\, 2 \mid 1 \,\} + 2 \,\middle|\, 4 \,\right\}\end{aligned}$$

In later sections, we will develop more techniques to quickly make sense of games that look like $\{\, x \mid y \,\}$ where x and y are numbers.

It can be easy to recognize integers, but cumbersome to justify them formally. Consider a position where one player is completely out of moves for the remainder of the game and the other player's remaining moves can be counted. In that case, we allow a shortcut for the analysis denoted with #:

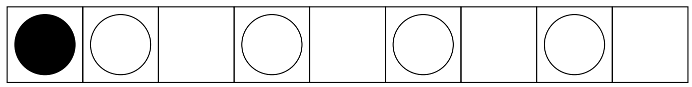
 $\overset{\#}{=} -3$

 $=^{\#} -3$

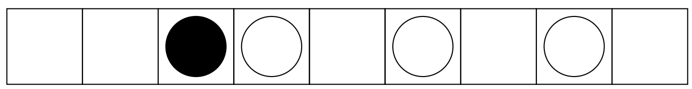
There are some important notes about this shortcut:

- This is not standard notation across other Combinatorial Game texts.
- This must be carefully employed. Only use this shortcut if the position definitely fits the requirements.

For example, the following KONANE position is not a simple position for counting, though upon first glance, it looks like it:

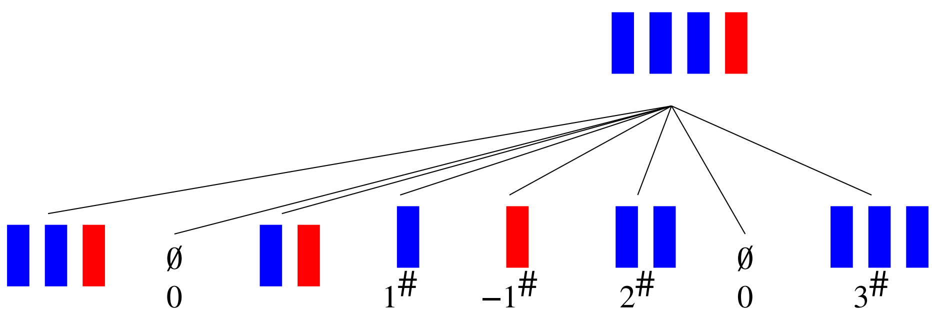
 $\neq 4$

This is because if L makes only the first jump to

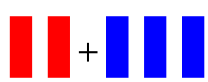


then R has an option.

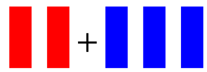
However, even if we can't do this on the root, we can do this on nodes further down a game tree like in the following TOPPLING DOMINOES position, with all simple countable integer options shown:



If players have moves in separate non-interactive components of the game, you can use this shortcut by listing out the sum of the integers of those components. E.g.:

 $=^{\#} \ (-2+3) = 1$

or

 $= (-2+3)^{\#} = 1$

Make sure to include both steps.

Computational Corner: Recognizing Values

Let's start writing a function to calculate the value of a position. How can we recognize the integers, or at least those we've seen so far?

- We can recognize zeroes.
- We can recognize (some) positive integers if the right options are empty.
- We can recognize (some) negative integers if the left options are empty.

We're going to just give up if we ever don't know what to do. Here's the code for handling when there is one right-hand option $\{ \mid x \}$:

```
def evaluate_position(position):
    '''Evaluates a game position, which is a list of two lists.
    The first list has left's options; the second has right's.'''
    left_options = position[0]
    right_options = position[1]
    if len(left_options) == 0 and len(right_options) == 1:
        # Case:1 Right option, 0 for Left.
        option = right_options[0]
        value = evaluate_position(option)
        if isinstance(value, int) and value > 0:
            #Not a winning move
            return 0
        elif isinstance(value, int):
            return value - 1
        else:
            raise Error('I don't know how to evaluate this!')
    elif len(left_options) == 1 and len(right_options) == 0:
        pass #saved for exercise  3.1.19
```

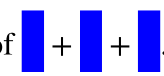
```
    elif len(left_options) == 0 and len(right_options) == 0:
        pass #saved for exercise  3.1.18
    else:
        raise Error('I don't know how to evaluate this!')
```

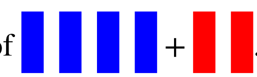
When you complete exercise 3.1.18 , you can test this out:

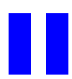
```
>>> zero = [[], []]
>>> negative_one = [[], [zero]]
>>> evaluate_position(negative_one)
-1
```

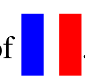
Because of the recursive case in here, evaluating a negative integer, $-n$, requires $O(n)$ time. When you get exercise 3.1.19 finished, you'll be able to evaluate (some) positive integers as well. (There are other forms that can also evaluate to integers, as we'll soon see!)

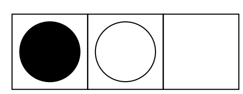
Exercises for 3.1

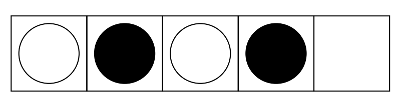
⋆ **0)** Find the integer value of . (Answer 3.1.0 in Appendix)

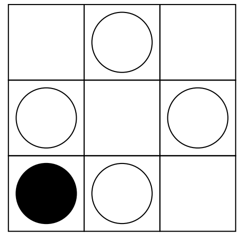
1) Find the integer value of .

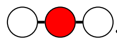
2) Prove that  equals 2.

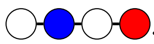
3) Find the value of .

⋆ **4)** What is the value of this KONANE position?  (Answer 3.1.4 in Appendix)

5) What is the value of this KONANE position? 

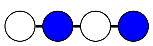
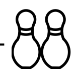
6) What is the value of this KONANE position? 

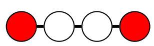
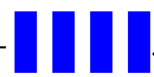
⋆ **7)** Find the integer value of  + . (Answer 3.1.7 in Appendix)

8) Find the integer value of  + .

⋆ **9)** Simplify $-1 + *3 + 4 + *$. (Answer 3.1.9 in Appendix)

10) Simplify $2 + *2 - 10 - *12$.

⋆ **11)** Find the value of  +  +  + . (Answer 3.1.11 in Appendix)

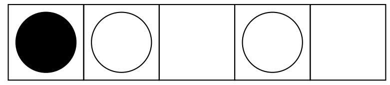
12) Find the value of  +  + .

⋆ **13)** Rewrite $\{\, 1 \mid 4 \,\} - 3$ as a single position and use basic arithmetic to simplify its options. (Answer 3.1.13 in Appendix)

⋆ **14)** Rewrite $\{\, 2 \mid -2 \,\} - 1$ as a single position and use basic arithmetic to simplify its options. Use this to determine the position's outcome class. (Answer 3.1.14 in Appendix)

15) Rewrite $\{\, -5 \mid -4 \,\} + 1$ as a single position and use basic arithmetic to simplify its options. Use this to determine the position's outcome class.

16) Rewrite $* + 4$ as a single position and use basic arithmetic to simplify its options. Use this to determine the position's outcome class.

17) Prove that KONANE position  is equal to 1.

⋆ **18)** Continue coding `evaluate_position` by implementing the bottom case: when there are no right or left options. (Answer 3.1.18 in Appendix)

19) Continue coding `evaluate_position` by implementing the second case: when there is one left option and no right options.

3.2. Simplest Numbers

If we have a game position where each player has a move to a number (e.g. $\{\, x \mid y \,\}$), so far we only know what to do if the option for both players is a move to zero. In that case, the game is just $*$.

Let's generalize this a bit further: what if we have $\{\, X \mid Y \,\}$ and $X < 0$ and $Y > 0$? Well, then L has only a move to $\mathcal{R}$ and R has only a move to $\mathcal{L}$. Since neither of these are winning moves, $\{\, X \mid Y \,\} \in \mathcal{P}$, and $\{\, X \mid Y \,\} = 0$.

What about the position $\{\, 0 \mid 2 \,\}$? It is in $\mathcal{L}$, so the value isn't 0.[2] Let's check whether it equals a positive game we're now familiar with: 1. Remember that to do this, we will find the outcome class of the difference, $\{\, 0 \mid 2 \,\} - 1$:

$$\begin{aligned}
\{\, 0 \mid 2 \,\} - 1 &= \{\, 0 \mid 2 \,\} + \{\ \mid 0 \,\} \\
&= \left\{\, 0 - 1 \;\middle|\; 2 - 1, \{\, 0 \mid 2 \,\} + 0 \,\right\} \\
&= \left\{\, -1 \;\middle|\; 1, \{\, 0 \mid 2 \,\} \,\right\}
\end{aligned}$$

L's only option is in $\mathcal{R}$, and both of R's options are in $\mathcal{L}$, so this sum is 0; $\{\, 0 \mid 2 \,\} = 1$.

Even more amazing, we don't really use the 2 at all. If we replace that with a bigger positive number, say 56, it doesn't change the result! Thus:

$$\{\, 0 \mid 2 \,\} = \{\, 0 \mid 3 \,\} = \cdots = \{\, 0 \mid 56 \,\} = \cdots = 1$$

[2] We are glad to see that the value of a game isn't one of its options!

What happens if we change the zero in $\{\, 0 \mid 56 \,\}$ to a 1? What is the value of $\{\, 1 \mid 56 \,\}$?

Based on the last answer, let's check to see whether it's equal to 2.

$$\begin{aligned}
\{\, 1 \mid 56 \,\} - 2 &= \{\, 1 \mid 56 \,\} + \{\ \mid -1 \,\} \\
&= \left\{\, 1-2 \;\middle|\; 56-2, \{\, 1 \mid 56 \,\} - 1 \,\right\} \\
&= \left\{\, -1 \;\middle|\; 54, \{\, 1 \mid 56 \,\} + \{\ \mid 0 \,\} \,\right\} \\
&- \left\{\, -1 \;\middle|\; 54, \left\{\, 1-1 \;\middle|\; 56-1, \{\, 1 \mid 56 \,\} + 0 \,\right\} \,\right\} \\
&= \left\{\, -1 \;\middle|\; 54, \left\{\, 0 \;\middle|\; 55, \{\, 1 \mid 56 \,\} \,\right\} \,\right\}
\end{aligned}$$

That last ugly option for $R\left(\left\{\, 0 \;\middle|\; 55, \{\, 1 \mid 56 \,\} \,\right\}\right)$ looks confusing, but all we care about is that L has a winning move to 0, so it is not in $\mathcal{R}$or $\mathcal{P}$. None of the options in the sum for either player are wins, so the sum is in $\mathcal{P}$, and the two parts are equal. Thus, $\{\, 1 \mid 56 \,\} = 2$. This leads us to the first part of a theorem known as the "Simplest Number Theorem":

If $G = \{\, x \mid y \,\}$, x and y are numbers, $x < y$, and there is an integer between x and y, then $G = z$, where z is the integer with smallest absolute value such that $x < z < y$.

Thus, we can simplify $\{\, -7 \mid -2 \,\}$ because:

- -7 and -2 are numbers,
- $-7 < -2$, and
- There are integers (-6, -5, -4, and -3) between them.

Of those, -3 has the lowest absolute value, so $\{\, -7 \mid -2 \,\} = -3$.

Important points:

- We don't have to worry about the cases where both z and $-z$ are between the two numbers, because then 0 will also be between them, and 0 is the integer with the smallest absolute value.

- We cannot use this if $x > y$. E.g. on $\{\, 5 \mid -3 \,\}$. (Note that this game is in $\mathcal{N}$, so it cannot have a number value.)

Computational Corner: Simplest Integer

Let's write a function to calculate the simplest integer, which should work like this:

```
>>> simplest_integer(5, 10)
6
>>> simplest_integer(-100, 0)
-1
>>> x = simplest_integer(-3, 1)
>>> print(x)
0
>>> x = simplest_integer(1, -3)
>>> print(x)
None
```

Although there are some sneaky tricks we can do to recognize simplest numbers without them, conditionals are probably the most readable solution here, even though they are a bit ugly. (We will assume we've already imported the math package.)

```
def simplest_integer(x, y):
  '''Returns the simplest integer between x and y.'''
  if not x < y:
      return None
  elif x < 0 and 0 < y:
      return 0
  elif x >= 0:
      simplest = math.floor(x+1)
      if simplest < y:
          return simplest
      else:
          return None
  else:
```

```
        simplest = math.ceiling(x-1)
        if simplest > x:
            return simplest
        else:
            return None
```

This function contains no recursive calls, no loops, and no complicated function calls; it runs in $O(1)$ time.

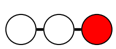
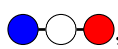
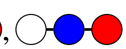
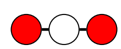
If there isn't an integer between x and y, we are not lost! In this case, we are going to look for halves, then fourths, then eighths, etc. Halves have the form $\frac{k}{2}$, fourths are $\frac{k}{4}$, and eights are $\frac{k}{8}$. Each time, the power of 2 in the denominator goes up. That means we are looking for a fraction like this where that power of 2 is minimized. More formally:

$$\{\, x \mid y \,\} = \frac{k}{2^j} \text{ where } x < \frac{k}{2^j} < y \text{ with the lowest possible } j$$

Numbers of the form $\frac{k}{2^j}$ (where k and j are integers, $j \geq 0$, and k is odd) are known as *dyadic rationals*.

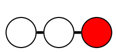
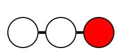
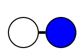
For example, if $G = \{\, 0 \mid 1 \,\}$, there are many dyadic rationals between 0 and 1, including $\frac{1}{4}$, $\frac{1}{2}$, and $\frac{3}{4}$ (and infinitely more). However, $\frac{1}{2}$ has the smallest denominator power of 2: 2^1. Thus, $\{\, 0 \mid 1 \,\} = \frac{1}{2}$.

Here is an example of a COL position with value $\frac{1}{2}$:

 $= \{$ ,  $\mid$  $\}$

$$= \{\, 0, -1 \mid 1 \,\}$$
$$= \{\, 0 \mid 1 \,\}$$
$$= \frac{1}{2}$$

This notion of fractions isn't very useful if our standard arithmetical addition doesn't work out. What happens if we add two halves together? Do we get 1? Let's check whether $\frac{1}{2} + \frac{1}{2} = 1$ by testing that $\frac{1}{2} + \frac{1}{2} - 1 = 0$:

 $+$  $+$  $= \{\, 0 \mid 1 \,\} + \{\, 0 \mid 1 \,\} + -1$

- L doesn't have a winning move in this because when they choose 0 in one of the $\{\ 0 \mid 1\ \}$ instances, R will pick 1 in the other, making the sum $0+1-1 = 0$.

- R can choose to play in one of the $\{\ 0 \mid 1\ \}$ pieces, in which case L will play in the other and win by making the sum 0 again. If they play in the -1, moving it to zero, that's even worse for them! L can move one of the $\{\ 0 \mid 1\ \}$s to a 0. R will have to make one of them 1 so that the sum is 1.

Since neither player has a winning move, the sum is zero, meaning that the game value $\frac{1}{2}$ acts just like we want it to in sums!

We have to be a bit careful here! Division does not (really) work with combinatorial game values. For example, just because $G + G = 1$, it is not necessarily true that $G = \frac{1}{2}$. We save the details for exercise 3.2.29 .

Let's look at fractional values with even larger denominators. Between 0 and $\frac{1}{3}$, there are many dyadic rationals, including $\left(\frac{1}{16}, \frac{1}{8}, \frac{3}{16}, \text{ and } \frac{1}{4}\right)$. $\frac{1}{4}$ is the "simplest" with the lowest denominator, so $\left\{\ 0 \;\middle|\; \frac{1}{3}\ \right\} = \frac{1}{4}$.[3]

Just like the integer case, we don't have to worry if there are two dyadic rationals with the smallest denominator. If that happens, then there is definitely another dyadic rational with a lower denominator. For example, if $\frac{5}{8}$ and $\frac{7}{8}$ are between the bounds, then so is $\frac{6}{8} = \frac{3}{4}$.

We can now combine these two cases for the full Simplest Number Theorem: If $G = \{\ x \mid y\ \}$, x and y are numbers, and $x < y$, then

- if there is an integer, z, such that $x < z < y$, then G is equal to the one with the lowest absolute value,

- otherwise, $G = \frac{k}{2^j}$ where $x < \frac{k}{2^j} < y$, k is an odd integer, and j is the smallest non-negative integer that fits the inequality.

We've now found games with values as small (absolute-value-wise) as any number we can want. In other words, no matter what positive number we can

[3]Note: we haven't yet seen a game with value $\frac{1}{3}$.

think of, we can always create a game between that number and zero. For any positive number, x, $\{\, 0 \mid x \,\}$ is also positive, and $\{\, 0 \mid x \,\} < x$.

Consider the position $\{\, 2 \mid \,\} + \{\, 0, * \mid 0, * \,\} = 3 + *2$. If it's L's turn to make a move, which should they choose? Their three options are: $2 + *2$, $3 + 0 = 3$, and $3 + *$. Remembering that all nimbers are confused with zero, we can see that both 3 and $3 + *$ are greater than $2 + *2$, so those two are the better options for L. (They are confused with each other, so neither is always better than the other.)

Let's look at another position, this time $\{\, 0 \mid 1 \,\} + \{\, 0 \mid 0 \,\} = \frac{1}{2} + *$, again considering that L is next to move. From L's two options ($*$ and $\frac{1}{2}$) we can compare and see that $* < \frac{1}{2}$, so L should move to $\frac{1}{2}$.

What about R on that same position? They could move to either $1 + *$ or $\frac{1}{2}$. $1 + * > \frac{1}{2}$, so it's in R's interest for them to move to $\frac{1}{2}$.

In all of these examples, it's better for the players to choose to move on the nimber part instead of the number. As it turns out, if there's a winning move for a player on the sum of a number and a non-number, it's always preferrable to play on the non-number part. How do we know? Let's state and prove this as a theorem!

Theorem 3.2.1 (Number Avoidance)**.** *Let $G + H \in \mathcal{L} \cup \mathcal{N}$ where G is an integer or dyadic-rational number and H is not a number. Then there is a left-option of H, J, such that $G + J$ is a winning move for L on $G + H$.*

In other words, if there is a winning move on that sum for L, there is a winning move on the non-number part. L might as well move on the non-number. Naturally, the same property holds true for R as well. We will prove this using induction.

Proof. Notice that all our numbers can be written with one of four forms: $\{\, x \mid y \,\}$, $\{\, \mid y \,\}$, $\{\, x \mid \,\}$, or $\{\, \mid \,\}$. The options (x and y) in all of those are also numbers. If we think about the L repeatedly making moves without any interaction from R, they will wind up in a position, $\{\, \mid y \,\}$ or $\{\, \mid \,\} = 0$, where they no longer have an option. In our proof, we will induct on the number of times L can do that.[4]

[4] The depth of this game is similar in idea to the *birthday* of a game, which we discuss in section 6.1.

Base Case: Let G be a number where L has no left-options. Since $G + H \in \mathcal{L} \cup \mathcal{N}$, L must have a winning move. Since there are no left-options on G, H must have a left-option J, such that $G + J \in \mathcal{L} \cup \mathcal{P}$. Thus, $G + J$ is a winning move for L. ✓

Inductive Assumption: Assume that for all integers and dyadic rationals G where L can make n moves in a row, but not $n+1$, and any non-number H where $G + H \in \mathcal{L} \cup \mathcal{N}$, H has a left-option, J such that $G + J$ is a winning move for L. To finish our proof, we need to show that this still works for G' instead of G where L can make $n + 1$ moves in a row on G.

Let G' be an integer or dyadic rational where L can make $n+1$ moves in a row, but not $n + 2$. Let H be a non-number such that $G' + H \in \mathcal{L} \cup \mathcal{N}$.

Let K be the left-option of G'. There are two cases. Either $K + H$ is a winning move for L or it is not.

Case 1: $K + H$ is not a winning move for L.

Then, just as in the base case, since $G' + H \in \mathcal{L} \cup \mathcal{N}$, it must have a winning move for L. Thus, there must be a left-option, J, of H such that $G' + J \in \mathcal{L} \cup \mathcal{P}$. ✓

Case 2: $K + H$ is a winning move for L.

Thus, $K + H \in \mathcal{L} \cup \mathcal{P}$. Since K is a number and H is not, $K \neq -H$, and $K + H \neq 0$. Thus, $K + H \in \mathcal{L}$. Since K L can make n moves in a row on K and $K + H \in \mathcal{L}$, it fits our inductive hypothesis. Thus H has a left-option, J such that $K + J \in \mathcal{L} \cup \mathcal{P}$, or $K + J \mathrel{|\!\triangleright} 0$. By Theorem 2.5.1, $G' \mathrel{|\!\triangleright} K$. Since G' and K are both numbers, $G' > K$. Thus, $G' + J > 0$, so $G' + J$ is a winning move for L from $G' + H$. ✓

Both cases hold, so the entire recursive case holds. ✓ □

The proof works for R just as it works for L.

This means that it's always better for a player to move on a non-number component of a sum rather than the number. Although so far we only know about a few non-numbers, we will see more later.

Exercises for 3.2

⋆ **0)** What is the single-number value of $\{\, 3 \mid 7 \,\}$? (Answer 3.2.0 in Appendix)

1) What is the single-number value of $\{\ -28 \mid -20\ \}$?

⋆ **2)** What is the single-number value of $\{\ 33 \mid 133\ \}$? (Answer 3.2.2 in Appendix)

3) What is the single-number value of $\{\ -10 \mid 10\ \}$?

⋆ **4)** List three dyadic rationals between $\frac{1}{8}$ and $\frac{7}{8}$. (Answer 3.2.4 in Appendix)

5) List three dyadic rationals between $-3\frac{1}{2}$ and -3.

⋆ **6)** What is the single-number value of $\{\ 10 \mid 11\ \}$? (Answer 3.2.6 in Appendix)

7) What is the single-number value of $\{\ -20 \mid -19\ \}$?

⋆ **8)** What is the single-number value of $\left\{\ -5\frac{1}{2}\ \middle|\ -5\ \right\}$? (Answer 3.2.8 in Appendix)

9) What is the single-number value of $\left\{\ -2\frac{3}{8}\ \middle|\ -2\frac{5}{16}\ \right\}$?

⋆ **10)** What is the single-number value of $\left\{\ -\frac{1}{2}\ \middle|\ \frac{1}{8}\ \right\}$? (Answer 3.2.10 in Appendix)

11) What is the single-number value of $\left\{\ 2\frac{1}{2}\ \middle|\ 4\ \right\}$?

⋆ **12)** Find an x and y such that $\{\ x \mid y\ \} = 9\frac{5}{8}$. (Answer 3.2.12 in Appendix)

13) Find an x and y such that $\{\ x \mid y\ \} = -1\frac{1}{4}$.

⋆ **14)** Find an x and y such that $\{\ x \mid y\ \} = -5\frac{11}{16}$. (Answer 3.2.14 in Appendix)

15) Find an x and y such that $\{\ x \mid y\ \} = 111\frac{3}{32}$.

⋆ **16)** What is the single-number value of $\left\{ \{ 2 \mid 6 \} \,\middle|\, \{ 6 \mid 10 \} \right\}$? (Answer 3.2.16 in Appendix)

17) What is the simplified value of $\left\{ \{ -3 \mid 1 \} \,\middle|\, \{ -1 \mid 3 \} \right\}$?

⋆ **18)** What is the simplified value of $\left\{ \{ 0 \mid 1 \} \,\middle|\, \{ 1 \mid 2 \} \right\}$? (Answer 3.2.18 in Appendix)

19) What is the simplified value of $\left\{ \{ -2 \mid \ \} \,\middle|\, \{ 1 \mid \ \} \right\}$?

⋆ **20)** What is the simplified value of $\left\{ \{ -10 \mid -1 \} \,\middle|\, \{ 0 \mid 4 \} \right\}$? (Answer 3.2.20 in Appendix)

21) What is the simplified value of $\left\{ \{ -100 \mid -99 \} \,\middle|\, \{ -98 \mid 0 \} \right\}$?

⋆ **22)** What is the simplified value of $\left\{ \left\{ \{ 3 \mid 5 \} \,\middle|\, \{ 11 \mid 13 \} \right\} \,\middle|\, \left\{ \{ 20 \mid 33 \} \,\middle|\, \{ 42 \mid 2176 \} \right.$

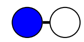
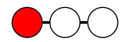
(The eight numbers used here were supplied by one of the authors' children.) (Answer 3.2.22 in Appendix)

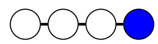
23) Use a direct proof to show that $\{ 0, * \mid 1 \}$ is equal to $\frac{1}{2}$.

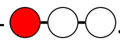
⋆ **24)** Use a direct proof to show that $\{ 0 \mid 1, 1{+}* \}$ is equal to $\frac{1}{2}$. (Answer 3.2.24 in Appendix)

25) We know that $\{ 0 \mid 1 \} = \frac{1}{2}$, and from exercise 3.2.24 , we know that $\{ 0 \mid 1, 1{+}* \} = \frac{1}{2}$ too. This doesn't work with $\{ 0 \mid 1{+}* \}$, however. Use a direct proof to show that it is instead equal to 1.

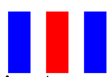
26) Determine which single-number value is equal to $\left\{ 1\frac{1}{2} \,\middle|\, \ \right\}$ and then use a direct proof to show that it is indeed the value. Hint: There are two likely candidates for the value, based on what you know about integers and simplest numbers.

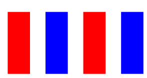
⋆ **27)** Find the number value of the sum of COL positions  + . (Answer 3.2.27 in Appendix)

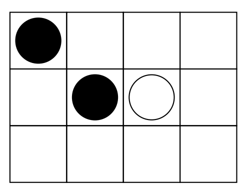
28) Find the number value of the COL position .

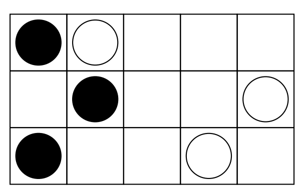
29) Let $G =$  + . Show that combinatorial game division doesn't work by proving two things:

- $G + G = 1$, and
- $G \neq \frac{1}{2}$

⋆ **30)** What is the value of  ? You may need to use the result from exercise 3.2.23 . (Answer 3.2.30 in Appendix)

31) What is the value of  ? You may need to use the result from exercise 3.2.30 .

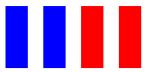
⋆ **32)** What is the value of this KONANE position:  ? (Answer 3.2.32 in Appendix)

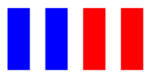
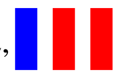
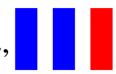
33) What is the value of this KONANE position:  ?

34) Prove that no integer or dyadic rational can be in $\mathcal{N}$.

3.3. Switches

Simplest numbers help us find that $\{ 0 \mid 2 \} = 1$, but it doesn't help us if we swap the two options: $\{ 2 \mid 0 \}$. These positions, $\{ x \mid y \}$ where x and y are numbers and $x \geq y$, are a new type of value: *switches*.

For example, here's a position you may encounter in TOPPLING DOMINOES: . What are the different options here?

 $= \Big\{ 0, 1, -2,$  $\Big|\ -1, 0, 2,$  $\Big\}$

The complicated positions are dominated on both sides (we save the proof of this for exercise 3.3.37) so the remaining options are:

$$\{ 0, 1, -2 \mid -1, 0, 2 \} = \{ 1 \mid -1 \}$$

As we mentioned before, this looks like a simplest-number case, except that $1 > -1$. Additionally, $\{ 1 \mid -1 \} \in \mathcal{N}$, so it can't be equal to zero. Just like nimbers, switches are values that are not numbers, so we will need separate notation for them.

As we said above, a switch is a position equal to $\{ x \mid y \}$, where $x \geq y$. We can rewrite this as $\{ b + h \mid b - h \}$ where:[5]

- $b = \frac{x+y}{2}$ and
- $h = \frac{x-y}{2}$

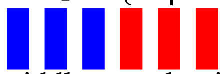
Since $x \geq y$, $h \geq 0$. Now, our switch notation will be to write $\{ b + h \mid b - h \}$ as $b \pm h$. We have names for these two pieces:

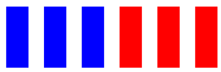
$$\underbrace{b}_{\text{base}} \pm \underbrace{h}_{\text{heat}}$$

Switches are extremely valuable for players to move on. They either provide moves for that player to use later or they deny their opponent moves. The *heat*

[5] We save the actual proof (using arithmetic) of this equivalence for exercise 3.3.38 .

of a switch tells the player how many moves they will gain by playing on it, i.e. how many moves that play is "worth". If the base of a switch is zero, we often drop the zero in front. For example, $\{\,3 \mid -3\,\} = 0 \pm 3 = \pm 3$.

Consider, for example, . If we (correctly) assume that the best moves are tipping the middlemost dominoes towards the opponents' dominoes, then we quickly see that

 $= \{\,2 \mid -2\,\} = \pm 2$

Now consider adding this to a COL $*$:

The reward for both players to move in the TOPPLING DOMINOES switch is extra moves for them to use later, while the COL position doesn't provide this.[6]

Our definition for a switch includes $* = \{\,0 \mid 0\,\} = \pm 0$.[7]

Since switches earn players extra moves to be used later, we will often refer to those extra moves as "points". Thus, for example, we would say that whoever plays on ± 3 "earns 3 points", independent of whether they are L or R.

What happens if we take the opposite (negative) of a switch with a base of zero? E.g. ± 3?

$$\begin{aligned} -\pm 3 &= -\{\,3 \mid -3\,\} \\ &= \{\,--3 \mid -3\,\} \\ &= \{\,3 \mid -3\,\} \\ &= \pm 3 \end{aligned}$$

Thus, $\pm x$ is its own switch![8]

When we add other values to zero-base switches, e.g. $*2 + \pm 7$, it sounds odd to say "plus plus minus", so we drop the operator and just write this as $*2 \pm 7$.

[6] As it turns out, there are no switches inside COL aside from $*$, which we can consider as ± 0.)

[7] This is not common across all definitions. Many texts will not include ± 0 as a switch.

[8] An easy proof of the general case is left for exercise 3.3.39 .

This works with numbers as well, so $-5 + \pm 3 = -5 \pm 3$. This value is exactly the same as the value $-5 \pm 3 = \{ -2 \mid -8 \}$.[9]

Let's look at something a little more complicated. If we have this sum of TOPPLING DOMINOES strands:

$G =$ 

Can we easily find the outcome class of G? How far can we simplify the value of this game?

With a reasonable guess at the dominating options and the values we've seen so far, we can quickly turn this into a sum of numbers and switches:

$$\begin{aligned} G &= \pm 3 - 1 \pm 1 \\ &= -1 \pm 3 \pm 1 \end{aligned}$$

The notation for a sum of switches like this is quite common, and it's normal to write them with the heats going from left to right in descending order. Thus, if we have two switches $b_1 \pm h_1$ and $b_2 \pm h_2$, and $h_1 > h_2$, then it's best to write the sum as:

$$b_1 \pm h_1 + b_2 \pm h_2 = (b_1 + b_2) \pm h_1 \pm h_2$$

It is actually not difficult to find the outcome class of a sum of switches. The best play (for both players) is always to move on the switch with the highest heat. That means that if L goes first in our TOPPLING DOMINOES example above, they will earn 3 points, then R will earn 1 point. After those two moves, the value will be $3 - 1 - 1 = 1$, a win for L. Using the same logic, R can earn $-3 + 1 - 1 = -3$. Both can win by moving first, so the game is in $\mathcal{N}$.

Is simplifying the value just as easy? Can we combine the two switch terms into one or cancel them out at all? Unfortunately, the answer is no. It might seem that we can condense $\pm 3 \pm 1$ into ± 2, except that the first position requires two moves to resolve the two switches and the second would need only one. $-1 \pm 3 \pm 1$ is as good as we're going to get.

[9] We save a proof of this for exercise 3.3.40 .

Okay, so perhaps there is a way to condense three such switches into a single term. Consider instead $\pm 7 \pm 4 \pm 1$. If players correctly played these from most beneficial to least, the first player would earn 7 for themselves, then the next would cancel out 4 of those, then the last would gain one back. Maybe this sum is equivalent to just ± 4. Let's try adding ± 5 to both and see what happens.

- In $\pm 4 \pm 5 = \pm 5 \pm 4$, the first player claims 5 points and the other player negates 4 by claiming the 4 for themselves. Whoever goes first gains a net of 1 point.

- In $\pm 7 \pm 4 \pm 1 \pm 5 = \pm 7 + \pm 5 + \pm 4 + \pm 1$, the first player will get 7, then lose 5 to the next player's move, then gain 4 again, then lose another 1. In total, whoever goes first gains a net of 5 points.

By adding a switch inbetween the terms of our sum, we see that we cannot collapse them as we might have wanted. The issue here is that we can have an expectation of the order those switches might be played on in their own group, but if we add another switch that falls somewhere inside that order, it will interrupt that expected order and change which points go to the first as opposed to the second player.

This occurs because no matter how close two different switches are, there can always be a switch between them. For example, $\pm 4\frac{1}{4} \pm 4$ is heavily changed by adding a switch inbetween, e.g. $\pm 4\frac{1}{4} \pm 4\frac{1}{8} \pm 4$.

Only in one obvious case can we simplify. Consider the position $\pm 3 \pm 3$. The result of this sum is each player getting 3 points, canceling each other out. It doesn't matter if this is added to other games, because the order of these two doesn't matter.

Thus, in general, $\pm x \pm x = 0$.

How do numbers compare with switches? Let's just consider zero-based switches. Using ± 4 as an example:

- Which numbers are less than ± 4?

- Which numbers are greater than ± 4?

- Which numbers are confused with ± 4?

Let's start by comparing with 4, which means we need to find $o(4 - \pm 4) = o(4 \pm 4)$.

$$\begin{aligned} 4 \pm 4 &= 4 + \{\, 4 \mid -4 \,\} \\ &= \{\, 8 \mid 0 \,\} \\ &\in \mathcal{N} \end{aligned}$$

Thus $4 \parallel \pm 4$. Notice that if our right option was anything larger, this would be in $\mathcal{L}$ instead. For example:

$$\begin{aligned} 4.01 \pm 4 &= 4.01 + \{\, 4 \mid -4 \,\} \\ &= \{\, 8.01 \mid .01 \,\} \\ &\in \mathcal{L} \end{aligned}$$

$4.01 > \pm 4$. We can add any positive number to that 4 to get a value greater than ± 4.

On the other side of zero, the same thing happens. $-4 \parallel \pm 4$, but $-4.001 < \pm 4$. When comparing to a number, x,

- $\pm 4 < x$ when $x < -4$,
- $\pm 4 \parallel x$ when $-4 \geq x \geq 4$, and
- $\pm 4 > x$ when $x > 4$.

This extends for any zero-based switch. When comparing $\pm k$ to a number x:

- $\pm k < x$ when $x < -k$,
- $\pm k \parallel x$ when $-k \geq x \geq k$, and
- $\pm k > x$ when $x > k$.

Exercises for 3.3

★ **0)** Rewrite $\{\ 2 \mid 0\ \}$ in the form $a \pm x$. (Answer 3.3.0 in Appendix)

1) Rewrite $\{\ 8 \mid 2\ \}$ in the form $a \pm x$.

★ **2)** Rewrite $\{\ -100 \mid -300\ \}$ in the form $a \pm x$. (Answer 3.3.2 in Appendix)

3) Following the example in the section, rewrite $\{\ 4 \mid 0\ \}$ in the form $a \pm x$. Use a direct proof to show that this equivalence holds *without* using the formulas for a and x presented earlier.

★ **4)** Rewrite the sum $\{\ 0 \mid 7\ \} + \{\ 7 \mid 0\ \}$ as a single position of the form $\{\ x \mid y\ \}$. (Answer 3.3.4 in Appendix)

5) Rewrite the sum $\{\ -1 \mid 2\ \} + \{\ 2 \mid -1\ \}$ as a single position of the form $\{\ x \mid y\ \}$.

★ **6)** Rewrite the sum $\{\ 2 \mid 5\ \} + \{\ 5 \mid 2\ \}$ as a single position of the form $\{\ x \mid y\ \}$. (Answer 3.3.6 in Appendix)

7) Rewrite the sum $\{\ 100 \mid 50\ \} + \{\ 50 \mid 100\ \}$ as a single position of the form $\{\ x \mid y\ \}$.

★ **8)** Rewrite the sum $\{\ -2 \mid -100\ \} + \{\ -100 \mid -2\ \}$ as a single position of the form $\{\ x \mid y\ \}$. (Answer 3.3.8 in Appendix)

9) Rewrite the sum $\{\ -2 \mid -3\ \} + \{\ 1 \mid 4\ \}$ as a single position of the form $\{\ x \mid y\ \}$.

★ **10)** Rewrite the sum $G = \{\ 6 \mid 2\ \} + \{\ -2 \mid -6\ \}$ as a sum of a number and switches and then find $o(G)$. (Answer 3.3.10 in Appendix)

11) Rewrite the sum $G = \{\ -8 \mid -18\ \} + \{\ 10 \mid 8\ \}$ as a sum of a number and switches and then find $o(G)$.

★ **12)** Rewrite the sum $G = \{ -1 \mid -2 \} + \{ 4 \mid 3 \}$ as a sum of a number and switches and then find $o(G)$. (Answer 3.3.12 in Appendix)

13) Rewrite $G = \pm 10 \pm 4$ without the $\pm$ symbol. (Hint: your position will have the form $\left\{ \{ a \mid b \} \,\middle|\, \{ c \mid d \} \right\}$.)

★ **14)** Rewrite $G = -2 \pm 3 \pm 1$ without the $\pm$ symbol. (Hint: your position will have the form $\left\{ \{ a \mid b \} \,\middle|\, \{ c \mid d \} \right\}$.) (Answer 3.3.14 in Appendix)

15) Rewrite $G = -10 \pm 9 \pm 8$ without the $\pm$ symbol.

★ **16)** Rewrite $G = 2\frac{1}{2} \pm 4 \pm 3$ without the $\pm$ symbol. (Answer 3.3.16 in Appendix)

17) Assuming a is a number, and b, and c are non-negative numbers, and that $b \geq c$, rewrite $G = a \pm b \pm c$ without the $\pm$ symbol.

★ **18)** Rewrite $G = \left\{ \{ 12 \mid 6 \} \,\middle|\, \{ 4 \mid -2 \} \right\}$ as a sum of two switches. (Answer 3.3.18 in Appendix)

19) Rewrite $G = \left\{ \{ -3 \mid -7 \} \,\middle|\, \{ -9 \mid -13 \} \right\}$ as a sum of two switches.

★ **20)** Rewrite $G = \left\{ \{ 4 \mid 3 \} \,\middle|\, \{ 2 \mid 1 \} \right\}$ as a sum of two switches. (Answer 3.3.20 in Appendix)

21) Rewrite $G = \left\{ \{ 2 \mid 2 \} \,\middle|\, \{ -4 \mid -4 \} \right\}$ as a sum of two switches.

★ **22)** Simplify $G = \left\{ \{ 2 \mid 4 \} \,\middle|\, \{ -2 \mid 0 \} \right\}$ to a switch or sum of switches. (Answer 3.3.22 in Appendix)

23) Simplify $G = \left\{ \{ 1 \mid 100 \} \,\middle|\, \{ 3 \mid -1 \} \right\}$ to a switch or sum of switches.

⋆ **24)** Simplify $G = \Big\{ \{ 14 \mid 50 \} \Big| \{ 4 \mid 100 \} \Big\}$ to a switch or sum of switches. (Answer 3.3.24 in Appendix)

25) Simplify $G = \Big\{ \{ 4 \mid \ \} \Big| \{ -10 \mid -6 \} \Big\}$ to a switch or sum of switches.

26) Simplify $G = \Big\{ \{ \ \mid -22 \} \Big| \{ -40 \mid -26 \} \Big\}$ to a switch or sum of switches.

⋆ **27)** $G = \Big\{ \{ 7 \mid 5 \} \Big| \{ -3 \mid x \} \Big\}$. Is it possible for G to be the sum of two switches? If so, find x. If not, explain why not. (Answer 3.3.27 in Appendix)

28) $G = \Big\{ \{ -11 \mid x \} \Big| \{ -21 \mid -31 \} \Big\}$. Is it possible for G to be the sum of two switches? If so, find x. If not, explain why not.

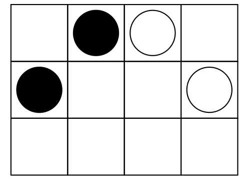
⋆ **29)** $G = \Big\{ \{ 1 \mid 0 \} \Big| \{ x \mid -2 \} \Big\}$. Is it possible for G to be the sum of two switches? If so, find x. If not, explain why not. (Answer 3.3.29 in Appendix)

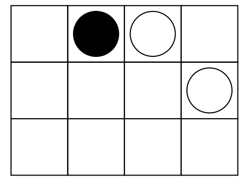
30) $G = \Big\{ \{ 6 \mid -2 \} \Big| \{ x \mid -4 \} \Big\}$. Is it possible for G to be the sum of two switches? If so, find x. If not, explain why not.

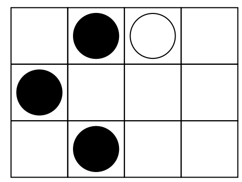
⋆ **31)** Simplify $G = \Big\{ \{ 11 \mid 15 \} \Big| \{ 7 \mid 15 \} \Big\} + \Big\{ \{ \ \mid -10 \} \Big| \{ -20 \mid -16 \} \Big\}$ by expressing it as a sum of switches. (Answer 3.3.31 in Appendix)

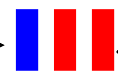
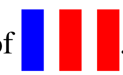
32) Simplify $G = \Big\{ \{ 7 \mid 10 \} \Big| \{ \ \mid -1 \} \Big\} + \Big\{ \{ \ \mid 0 \} \Big| \{ -8 \mid -4 \} \Big\}$ by expressing it as a sum of switches.

33) Simplify $G = \Big\{ \{ 11 \mid 21 \} \Big| \{ -21 \mid -7 \} \Big\} + \Big\{ \{ 17 \mid 21 \} \Big| \{ -21 \mid -1 \} \Big\}$ by expressing it as a sum of switches.

★ **34)** What is the value of this KONANE position:  ? (Answer 3.3.34 in Appendix)

35) What is the value of this KONANE position:  ?

36) What is the value of this KONANE position:  ?

37) In the chapter, we skipped the step showing that $1 >$ . Prove this by finding the value of .

38) Use some arithmetic in a direct proof to show the assertion in the section, that if $\{\, x \mid y \,\} = \{\, b+h \mid b-h \,\}$, then $b = \frac{x+y}{2}$ and $h = \frac{x-y}{2}$.

★ **39)** Use a direct proof to show that $- \pm x = \pm x$ for any non-negative number x. (Answer 3.3.39 in Appendix)

40) Use a direct proof to show that $n + 0 \pm h = n \pm h$ for any number n. Hint: This proof is extremely short if you use Number Avoidance!

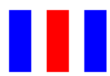
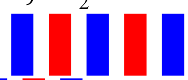
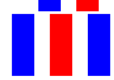
★ **41)** Consider $G = \left\{ \{\, w \mid x \,\} \,\middle|\, \{\, y \mid z \,\} \right\}$ where w, x, y, and z are all numbers. Using your answer from exercise 3.3.17 , fill in the boxes below with comparison operators to give four necessary and sufficient conditions for G to be the sum of two switches:

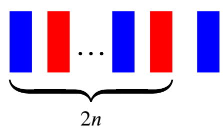
- $w \square x \square y \square z$ and
- $w - x \square y - z$

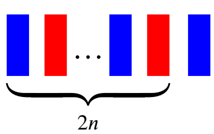
(Answer 3.3.41 in Appendix)

42) Use your answer to exercise 3.3.17 to write a direct proof that your answers to exercise 3.3.41 are correct. (In other words, show that these conditions are necessary for G to be the sum of two switches.)

3.4. Other rational game values

As we saw in Section 3.2, the only rational values that a short game can achieve are dyadic rationals, i.e. $\frac{p}{2^n}$ for some integers p and n. As a short refresher, note that the TOPPLING DOMINOES position  equals $\{\, 0 \mid 1 \,\} = \frac{1}{2}$ since L can move to 0 and R can move to 1. Next, consider the position . Here, L's best move is still to 0, and now R's best move is to . Hence, this position has value $\left\{ 0 \,\middle|\, \frac{1}{2} \right\} = \frac{1}{4}$.

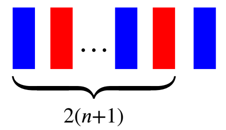
Continuing in this way, it seems that  (n copies of alternating pairs and ending with a blue domino) has value $\frac{1}{2^n}$. Let's prove this using mathematical induction.

Claim 3.4.1.  *has value* $\frac{1}{2^n}$ *for all* $n \geq 0$.

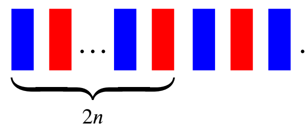
Proof. We proceed by induction on n.

Base Case: $P(0)$. We've chosen the smallest possible value for our base case. This is simply the position  which has value $1 = \frac{1}{2^0}$. So the base case is true. ✓

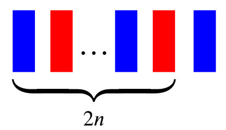
Inductive Assumption: Assume $P(n)$ is true. Consider the position



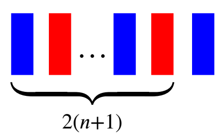
This is the same as



We can see that L's best move is to clear the board and move to 0, while R can, at best, move to the position

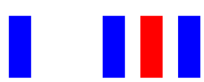


By our inductive assumption, we know that this position has value $\frac{1}{2^n}$. In fact, we know this is the best move for R because, by our inductive assumption, all other options are greater. Therefore,

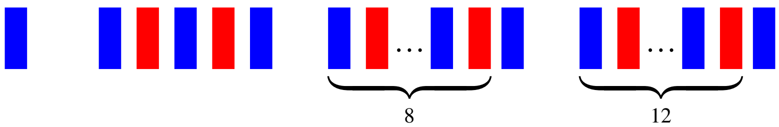


has value $\left\{ 0 \;\middle|\; \frac{1}{2^n} \right\} = \frac{1}{2^{n+1}}$. Hence the claim is true. $\square$

We have found a TOPPLING DOMINOES position for every positive integer power of $\frac{1}{2}$! Let's use this construction to find our first game with a rational value that isn't dyadic. We know that we can add values of TOPPLING DOMINOES positions by simply lining them up with a space between them. For example,

 has value $1 + \frac{1}{2} = \frac{3}{2}$ and

has value $1 + \frac{1}{2} + \frac{1}{4} = \frac{7}{4}$. You'll recall from Calculus that $\sum_{k=0}^{\infty} ar^k$ converges to $\frac{a}{1-r}$ whenever $|r| < 1$. So what if we consider an infinitely long strip composed of smaller and smaller valued positions?



has value $1 + \frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \ldots = \sum_{k=0}^{\infty} \frac{1}{4^k} = \frac{1}{1-\frac{1}{4}} = \frac{4}{3}$, which is not dyadic. Therefore, it may seem that if we consider games with infinite size we are no longer restricted to dyadic rational game values. However, we need to be careful. If our game really has value $\frac{1}{3} > 0$, then L has a winning move. However, this game will not end in a finite number of moves, so it can't be in $\mathcal{L}$. Infinite games unfortunately require a lot of machinery we aren't ready to handle yet, and are outside the scope of this book.

3.5. Infinitesimals

So far we have seen games with rational values that are greater than, less than, or equal to zero. We have also seen fuzzy games that are incomparable to zero. But we can also find games that are greater than zero but smaller than anything we've seen so far.

Consider the following Toppling Dominoes position:

A game tree will demonstrate that L's best option is 0 and R's best option is $*$. For now, let $u = \left\{ 0 \mid * \right\}$ represent this game value. Let's prove directly that $u < r$ for any positive rational number r.

Proof. Let $r \in \mathbb{Q}^+$. There is some $n \in \mathbb{N}$ such that $1/2^n < r$. Let's examine the game

$$\begin{aligned} G &= 1/2^n - u \\ &= 1/2^n + \left\{ * \mid 0 \right\} \end{aligned}$$

By Theorem 3.2.1 (*Number Avoidance Theorem*), if L has a winning move then it is to $1/2^n + *$. Then, similarly, R can at best play to $1/2^n \in \mathcal{L}$. If instead R moves first, it is to $1/2^n \in \mathcal{L}$. Therefore, $G \in \mathcal{L}$ and the game $u < r$ for any positive rational number r. □

We have found a positive game value that is smaller than every rational number! We call this value an *infinitesimal*, and denote u by the symbol $\uparrow$. Similarly, we denote $-u$ by $\downarrow$, which is an infinitesimal in $\mathcal{R}$. We will use the term infinitesimal to refer to any non-zero value between all negative rationals and all positive rationals. So any nimber is also an infinitesimal.

Claim 3.5.1. *If $n \in \mathbb{N}^+$ then $*n$ is an infinitesimal.*

Proof. Let $r \in \mathbb{Q}^+$. It is sufficient to show that $-r <*n < r$. First, note that in the game $r - *n$, which is equivalent to $r + *n$, L can move to r, and R can only move to $s - *n (= s + *n)$ or $r - *k (= r + *k)$ for some $s > r, k < n$. Since $s > r, s + *n > r + *n$. In the latter case, L can move to $r > 0$. So $*n < r$. We leave the remainder of the proof to the reader. □

Comparisons can get weird with infinitesimals. For example, $\uparrow > 0$, but both $\uparrow$ and 0 compare oddly with the sum $\uparrow + *$.

$\uparrow + * = \left\{ 0 \,\middle|\, * \right\} + \left\{ 0 \,\middle|\, 0 \right\}$. L wins by making their move on $*$; R wins by making their move on $\uparrow$. Thus, $\uparrow + * \in \mathcal{N}$, so $\uparrow + * \parallel 0$.

Now let's compare $\uparrow$ and $\uparrow + *$ by finding the outcome class of $\uparrow -(\uparrow + *)$:

$$\begin{aligned} \uparrow -(\uparrow + *) &= \uparrow - \uparrow - * \\ &= \uparrow + \downarrow + * \\ &= 0 + * \\ &= * \in \mathcal{N} \end{aligned}$$

Thus, $\uparrow \parallel \uparrow + *$. We have found three games, G, H, and I where $G \parallel H \parallel I$, but $G \nparallel I$! In fact, $\uparrow + *$, also written $\uparrow *$, demonstrates some weird properties of nimbers. Let's compare it to a few of them. Remember that every nimber is its own inverse, so subtracting a nimber is equivalent to adding it.

0: We have just seen that $\uparrow * \parallel 0$.

*: $\uparrow * - * = \uparrow > 0$

$*2$: $\uparrow * - *2 = \uparrow + * + *2 = \uparrow + *3$
If L goes first, they can move to $\uparrow > 0$.
If R goes first, they can move to either $* + *3 = *2$, $\uparrow$, $\uparrow *$, or $\uparrow + *2$, all of which are positive or confused with 0. Hence R loses moving first, and $\uparrow * - *2 > 0 \Rightarrow \uparrow * > *2$.

$*k$: A similar argument to above shows that $\uparrow * > *k$ whenever $k > 2$.

We've just seen that there is a game, $\uparrow *$, that is bigger than $*$ but confused with 0 and every other nimber. You'll see in the exercises that there is a game g bigger than $*k$ and confused with every other nimber. Hence, there is no natural order to the nimbers whatsoever.

Before proceeding, let's introduce a new concept and a result that will help us.

Consider playing a game G in isolation. That is, two players on a single game, alternating turns. If L moves first and both players play optimally, then the first number that results is called the *left stop* of the game, denoted $L(G)$. The *right stop*, $R(G)$ is the first number reached if instead R moves first. We can define the stops of a game using the following recurrence relation.

$$L(G) = \begin{cases} G & G \text{ is a number} \\ \max\limits_{\forall G^L} R(G^L) & \text{otherwise} \end{cases} \qquad R(G) = \begin{cases} G & G \text{ is a number} \\ \min\limits_{\forall G^R} L(G^R) & \text{otherwise} \end{cases}$$

For any game G we can see that $L(G) \geq R(G)$. What about stops in infinitesimals? We present the following theorem without proof, as its proof is outside the scope of this course.

Theorem 3.5.2. *G is infinitesimal if and only if $L(G) = R(G) = 0$.*

We will see a larger infinitesimal in Section 6.1. For now, let's see what other infinitesimal values we can find. Since $\uparrow = \{ 0 \mid * \}$, what about the game $g = \{ 0 \mid \uparrow \}$?

We can see that $g > 0$. How does it compare to $\uparrow$?

$$\begin{aligned} g - \uparrow &= g + \downarrow \\ &= \{\, 0 \mid \uparrow \,\} + \{\, * \mid 0 \,\} \\ &= \left\{\, 0 \,\middle|\, \{\, 0 \mid * \,\} \,\right\} + \{\, * \mid 0 \,\} \end{aligned}$$

From here, L can move to $\downarrow < 0$ or to $g + *$, and R can move to g or to $\uparrow + \downarrow = 0$. Since R has 0 as an option, we know that $g - \uparrow \notin \mathcal{L}$. Hence, $g \not> \uparrow$. It can be similarly shown that $\uparrow - g \notin \mathcal{L}$, hence $g \not< \uparrow$. So the game g is incomparable to $\uparrow$. In fact, $g = \uparrow + \uparrow + *$, which we denote $\Uparrow *$.

We can find much smaller values than $\uparrow$ and $\Uparrow *$. Consider the game $r = \{\, 0 \mid -1 \,\}$ and another game $+_1$ in which L can move to 0 but R can only move to r (so $+_1 = \left\{\, 0 \,\middle|\, \{\, 0 \mid -1 \,\} \,\right\}$). We can see that $r \in \mathcal{N}$, though it also, in some sense, is a bit better for R than for L. However, the game $+_1 \in \mathcal{L}$ since no matter which player moves first, L has a winning strategy. The game $+_1$ is read *tiny 1*. In fact, we can replace 1 in r with any game G to yield $+_G = \left\{\, 0 \,\middle|\, \{\, 0 \mid -G \,\} \,\right\}$. Naturally, the game $-+_G = \left\{\, \{\, G \mid 0 \,\} \,\middle|\, 0 \,\right\}$ is called *miny* G and is written $-_G$. Let's examine some tiny and miny positions.

First, we prove that if $G > 0$, then $+_G$ is infinitesimal and positive.

Proof. Let $G > 0$ be a game. In $+_G$, L can move to 0, while R moving first can move to a position in $\mathcal{N}$. So $+_G \in \mathcal{L}$.

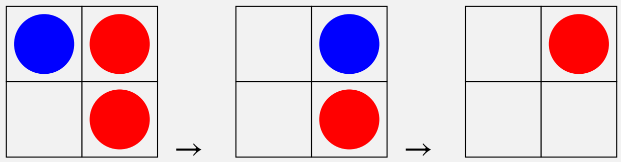
Now, to show that it's infinitesimal, we apply Theoremm 3.5.2 and consider the stops of G. When L moves first, the game ends at 0 in one move. If R moves first, then the game moves to $\{\, 0 \mid -G \,\}$, after which L moves to 0. Since both stops are 0, the game $+_G$ is infinitesimal.

□

Let's examine an ruleset that has some interesting properties related to infinitesimals.

CLOBBER

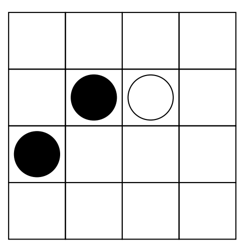
CLOBBER is a partisan game in which players take turns moving a piece of their color in any of the four orthogonal directions onto a piece of their opponent's color, removing the other's piece from play.



L moves right to take the red piece, then R moves up to win.

Note that, in CLOBBER, if one player has a move then so does the other. In the game tree of any position in CLOBBER, therefore, every node has a sibling. This is an example of a *dicotic* game. What values do you think are possible in a dicotic game? Note that a positive (or negative) integer is not dicotic, which is evident from its game tree. Since any option of a dicotic game is also dicotic, we have that no integer can be an option of a dicotic game. Since integers appear as values in the game tree of any rational number, we exclude all rational values. In fact, every dicotic game attains only infinitesimal values. Such a game is called *all-small*.

Exercises for 3.5

0) What is the value of this KONANE position:  ?

1) Complete the proof that if $n \in \mathbb{N}^+$ then $*n$ is an infinitesimal.

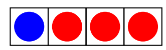
⋆ **2)** Since $\uparrow > 0$ we know that $G + \uparrow > G$ for any game G. So, in particular,

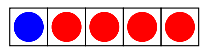
$\uparrow + \uparrow > \uparrow$. Prove that $\uparrow + \uparrow$, denoted $\Uparrow$ is an infinitesimal. (Answer 3.5.2 in Appendix)

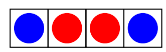
3) Perhaps surprisingly, the sum $\uparrow + \uparrow + *$ is equivalent to $\{\, 0 \mid \uparrow \,\}$. Prove that this is the case. ($\uparrow + \uparrow$ is usually written as $\Uparrow$)

⋆ **4)** Show that the game $+_1 + *k$ is bigger than $*k$. (Answer 3.5.4 in Appendix)

5) Show that the game $+_1 + *k$ is confused with $*j$ whenever $j \neq k$.

⋆ **6)** What is the value of the CLOBBER position  ? (Answer 3.5.6 in Appendix)

7) What is the value of the CLOBBER position  ?

8) What is the value of the CLOBBER position  ?

9) Determine whether or not $||$ is an equivalence relation. Hint: We're in the section on infinitesimals!

10) Determine whether or not $\triangleleft|$ is a partial order.

4. Strategy

Let's revisit the questions we started off with. Remember that we're considering these when it is our turn in a combinatorial game.

1. Is it possible for us to win this game?
2. Can we win no matter what our opponent does?
3. Which of our possible moves is part of a winning strategy?

Let's discuss each of these and see apply everything we've learned to try to answer them.

4.1. They-Love-Me-They-Love-Me-Not

The first question, "Is it possible for us to win this game?" avoids the assumption of optimal moves by the two players. Instead, we are asking whether there is *any* back-and-forth path through the game tree that results in a winning outcome. This seems likely, but there is a whole family of positions where the winner is independent of which moves are made; neither player can change the outcome by making good (or bad) moves. For example, consider a NIM position with multiple piles of only one stick.

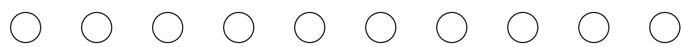
/ / | / \ \ |

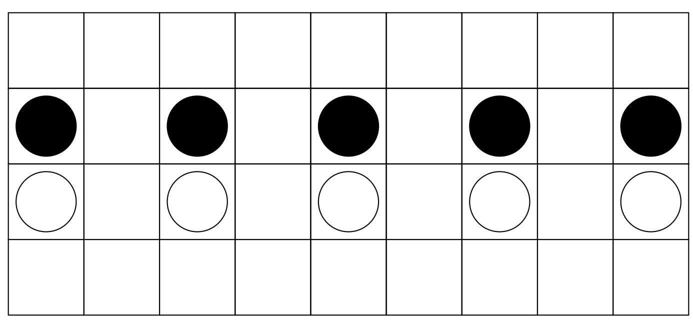
The value of these kinds of positions is always only 0 or $*$; there are no strategic decisions to be made.

These situations are known as *They-Love-Me-They-Love-Me-Not* (short: TLMTLMN). This name is a reference to the originally-French pasttime, "He (She) Loves Me, He (She) Loves Me Not". In this activity, the person alternatively speaks the two phrases "He (She) Loves Me" and "He (She) Loves Me Not", each time simultaneously plucking one petal off a flower[1]. Whichever phrase is uttered as the last petal is removed is supposed to represent how the object of affection of the "player" feels. Cartoons often depict the player as exhibiting back-and-forth assumptions of how the game will end with each petal, even though the parity of the number of petals remaining becomes more and more obvious[2].

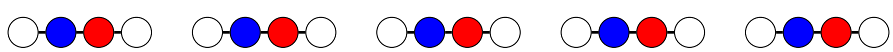
Instead of alternating phrases about states of amour, the phrases more appropriate for combinatorial game positions are: "I Win, They Win". We imagine that on each of their turns, the player says to themselves, "I win" and says "They win" during their opponent's turn. If there is nothing strategic the players can do other than take their turns, then the outcome is only dependent on the parity of the number of turns remaining.

They-Love-Me-They-Love-Me-Not situations are not restricted to impartial rulesets. This Col position and Konane position are both They-Love-Me-They-Love-Me-Not:



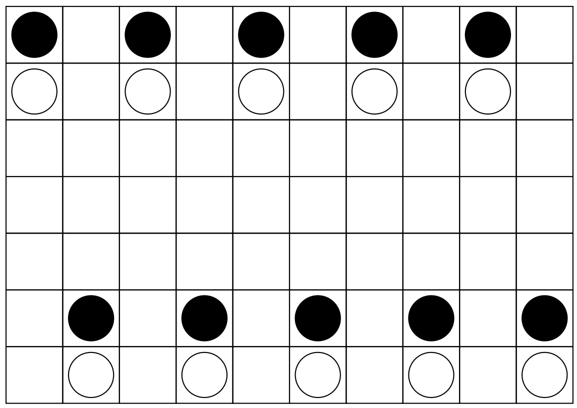


Nor do the positions need to be alternating between $*$ and 0. These Col and Konane positions will alternate between either 1 and 0, or -1 and 0, depending on who goes first:



[1] The flower is commonly depicted as an ox-eye daisy.

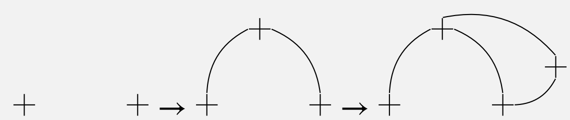
[2] Here is an example from Disney's *The Little Mermaid*: `https://youtu.be/Hqv1m4Gsfmw`.



In the most general form, a They-Love-Me-They-Love-Me-Not position is one where all possible game paths have the same parity of length. Some rulesets contain only TLMTLMN positions. A common example of this is BRUSSELS SPROUTS:

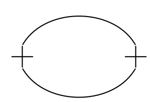
BRUSSELS SPROUTS

BRUSSELS SPROUTS is an impartial game played on a planar graph. Each vertex has four "arms" where edges can be attached. Each arm can only connect to one edge. A turn consists of creating a new vertex (with the four arms) then adding edges from two opposite arms of that new vertex to connect to previously-created arms. These new edges must maintain the planarity of the graph, which means that when drawn on paper, they may not cross any edges nor vertices. The chosen arms must not yet have attached edges. If there is no way to draw a new vertex with such arms, then there are no legal moves.
For simplicity of actual play, instead of drawing circles for the nodes, it is common to draw a "plus sign" to indicate the four arms, or just draw a short line segment across a line to signify the new node on that edge.

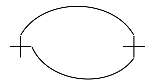


The first player adds the node at the top and draws the two edges. The second player, then connects that same node with the one on the right. The new node that's drawn can't connect on the left-hand-side.

Consider this BRUSSELS SPROUTS position, one move after starting from a single-node empty board:



The value of this game is 0, as there are exactly two moves from this position. (You can connect the outer arms and the inner arms, and then no more connections can be made.) In fact, from the initial position with just a single node (+) there are only two different (non-symmetric) options. Either connecting two opposite arms, or two adjacent arms, like this:



It might seem that there is some strategic choice to be made between these two, but this second option is also a zero.

Not only is there not a strategic choice to make here, in BRUSSELS SPROUTS, there are *never* strategic choices! All options of each position have the same value! Thus, each position is either $*$ or 0. This is because from a starting position with k nodes, there will always be exactly $5k - 2$ moves to reach the end of the game.

This is not an obvious fact; proving it requires something called the *Euler Characteristic* of planar graphs. The Euler Characteristic is a property of a connected graph, specifically $|V| - |E| + f$, where:

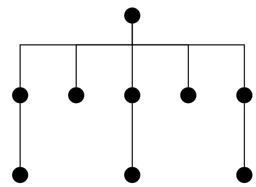
- V is the set of vertices, so $|V|$ is the number of vertices (the size of that set),
- E is the set of edges, so $|E|$ is the number of edges, and
- f is the number of *faces*. In planar graphs, this is the number of distinct regions, separated by edges. (The space outside the graph is one region.)

It is known that the Euler Characteristic of a (connected) planar graph is always 2. We can also express $|V|$, $|E|$, and f in terms of the number of moves to finish the game and number of initial nodes. (We save this proof for exercises 4.1.7 and 4.1.8 .)

It is common for games of some rulesets to breakdown into a bunch of disjoint one-move pieces, so it is helpful to recognize when this is you are approaching one of these situations.

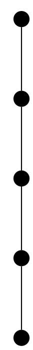
Exercises for 4.1

⋆ **0)** Is the following impartial game tree a They-Love-Me-They-Love-Me-Not (TLMTLMN) position? Find the outcome class and the value of the tree.

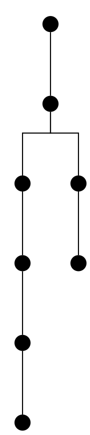


(Answer 4.1.0 in Appendix)

1) Is the following impartial game tree a They-Love-Me-They-Love-Me-Not (TLMTLMN) position? Find the outcome class and the value of the root of the tree.



2) Is the following impartial game tree a They-Love-Me-They-Love-Me-Not (TLMTLMN) position? Find the outcome class and the value of the root of the tree.

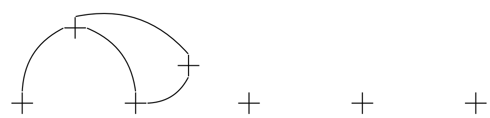


3) Let G be a position where all followers have value either $*$ or 0. Prove that G is TLMTLMN by proving that all leaves must have the same parity of distance from G.

⋆ **4)** What is the value of a BRUSSELS SPROUTS starting position with one node? (Answer 4.1.4 in Appendix)

5) What is the value of a BRUSSELS SPROUTS starting position with four nodes?

6) What is the value of this BRUSSELS SPROUTS position?



7) What are the values of $|V|$, $|E|$, and f at the end of a BRUSSELS SPROUTS game in terms of k, the number of initial nodes, and m, the number of moves played? Explain each of your answers. Hint: f is the trickiest!

8) Use your answers to Exercise 4.1.7 to prove that the number of moves from the starting position will always be $5k - 2$.

4.2. Determining Winnability

The second question, "Can we win no matter what our opponent does?", is really just asking what the outcome class is. If we know the outcome class and we know who plays next, we know which player has a winning strategy, so we can answer the question.

Everything we've learned so far can help us to find the outcome class. As we mentioned back in section 0.7, it's often easier to find the values of individual game components and add them together than it is to find the overall outcome class based on the game tree of the sum of those components.

So, to determine winnability, we can consider this algorithm:

1. Separate the position into separate, disjoint components.

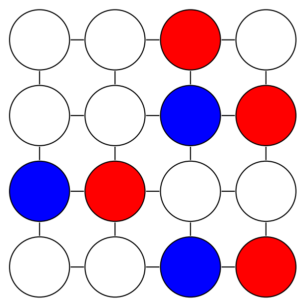
2. Find the values of each of those components.

3. Add those values together to get the overall value.

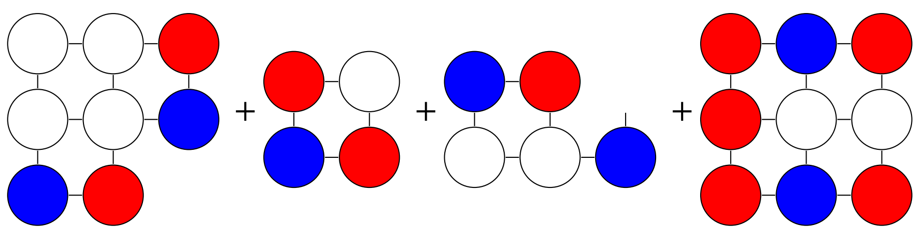
4. Determine which outcome class that value is in.

5. Use the outcome class to decide whether you have a strategy to win!

Steps 2 and 4 are often the most difficult. For the second step, you'll have to enumerate the options for each player, then recursively find those values of those positions. They may decompose into independent parts on their own. That will make evaluation easier, so you would definitely try that before finding their individual values. Consider this COL example:

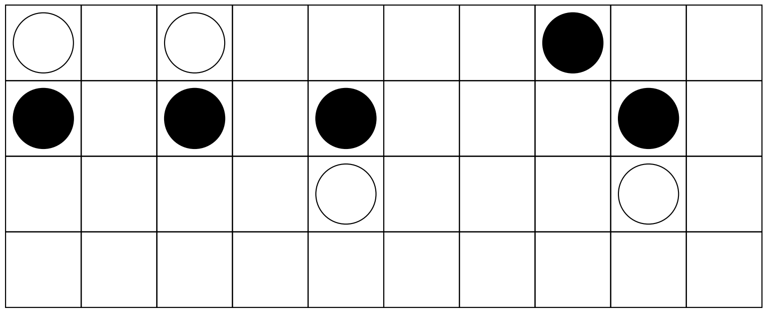


This breaks down into four independent positions:

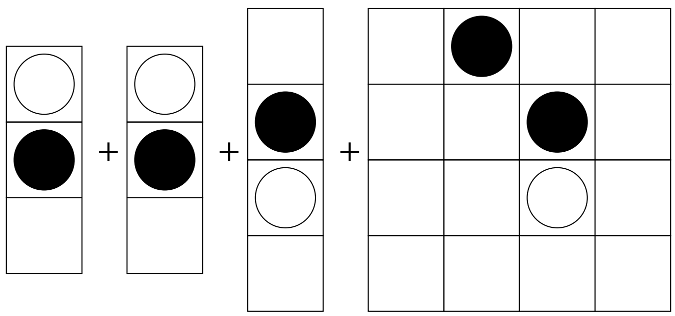


The values of the parts are 0, 1, -1, and 1, respectively, so the total value is 1. $1 \in \mathcal{L}$, so L has a winning strategy on the overall position no matter who goes first.

The value of the game could be more complex. Let's consider a KONANE position:



This also decomposes into four independent component positions:



These components have values -1, -1, $*$, and $\uparrow$, so the total value is $-2+\uparrow + *$. Since $\uparrow$ and $*$ are infinitesimals, any (finite) sum of them will be lower than the absolute value of any number. Thus, $|-2| > \uparrow + *$, meaning that $-2+\uparrow + * < 0$, so the sum is in $\mathcal{R}$.

If our sum, G, consists of sums of numbers, arrows, and nimbers, we can keep those three parts separate. The numbers sum to a single number and the nimbers sum to a single nimber. Thus, the total will look like:

$$G = \underbrace{x}_{\text{number}} + a\uparrow + *k$$

To determine the outcome class of this sum, we can follow this procedure:

1. If $x \neq 0$, then the arrows and nimbers don't matter. The outcome class is wherever the number resides.

2. If $x = 0$ and $a = 0$ and $k = 0$, then $G \in \mathcal{P}$.

3. If $x = 0$ and $a = 0$, but $k \neq 0$, then $G \in \mathcal{N}$.

4. If $x = 0$, but $a \neq 0$, we can't just use a to determine the outcome class. The reason for this is that some arrows are incomparable with some nimbers. E.g. $\uparrow \| *$.

Luckily, the overlap in confusion between arrows and nimbers is not gigantic. $\uparrow \| *$, but the same is not true of other sums of arrows and nimbers and $*$. For example, $\uparrow + * + * = \uparrow \in \mathcal{L}$, so $\uparrow + * > *$; these two are not confused with each other. Let's check $\Uparrow$ and $*$:

$$\begin{aligned} \Uparrow + * &= \{\, 0 \mid * \,\} + \{\, 0 \mid 0 \,\} \\ &= \{\, \uparrow + *, \Uparrow \mid \uparrow + * + *, \Uparrow \,\} \\ &= \{\, \uparrow + *, \Uparrow \mid \uparrow, \Uparrow \,\} \\ &= \{\, \uparrow + *, \Uparrow \mid \uparrow \,\} \\ &\in \mathcal{L} \end{aligned}$$

Thus, $\Uparrow > *$.

Let's check how $\uparrow$ compares to $*2$:

$$\begin{aligned} \uparrow + *2 &= \{\, 0 \mid * \,\} + \{\, 0, * \mid 0, * \,\} \\ &= \{\, *2, \uparrow, \uparrow + * \mid *3, \uparrow, \uparrow + * \,\} \\ &\in \mathcal{L} \end{aligned}$$

L has winning moves and R does not in the sum (difference), so $\uparrow > *2$. As it turns out, the only confusion between arrows and nimbers is $\uparrow \| *$. [3]

Thus, looking back at $G = x + a \uparrow + * k$, if $x = 0$ and $a \neq 0$, we can use a to determine the outcome class except when $G = \uparrow + *$ or $G = \downarrow + *$. In those two cases, $G \in \mathcal{N}$. In all other cases, $G \in o(a)$. Let's update our categories from above:

[3] You guessed it; we're saving the proof of this for an exercise! Namely, exercise 4.2.2 .

1. If $x \neq 0$, then the arrows and nimbers don't matter. The outcome class is wherever the number resides.

2. If $x = 0$ and $a = 0$ and $k = 0$, then $G \in \mathcal{P}$.

3. If $x = 0$ and $a = 0$, but $k \neq 0$, then $G \in \mathcal{N}$.

4. If $x = 0$, $k = 1$, and $|a| = 1$, then $G \in \mathcal{N}$.

5. If $x = 0$, and either $k \neq 1$ or $|a| > 1$, then $G \in o(a)$ (meaning that if $a > 0$, then $G \in \mathcal{L}$ and if $a < 0$, then $G \in \mathcal{R}$).

Now what happens if we include switches in G? How does this change how we evaluate the outcome class? Before we consider the whole thing, let's look at the outcome class of a sum of switches:

$$G = \underbrace{x}_{number} \pm s_1 \pm s_2 \pm \cdots \pm s_n$$

Recall that for a single switch $G = x \pm s = \{ x + s \mid x - s \}$:

- $G \in \mathcal{N}$, if $|x| \leq s$
- $G \in \mathcal{L}$, if $x > s$
- $G \in \mathcal{R}$, if $x < -s$

With two switches, $G = x \pm s_1 \pm s_2$, we can see what will happen after two turns. The first player will take s_1 and the second will take s_2. If L goes first, the value will become $x + s_1 - s_2$ when it is L's turn again. If R goes first, it will instead be $x - s_1 + s_2 = x - (s_1 - s_2)$ at the start of their next turn.

Let's look more closely at the two cases based on who goes first.

- If L goes first, they will win if and only if $x + s_1 - s_2 > 0$, or $-x < s_1 - s_2$. In other words, R wins if and only if $-x \geq s_1 - s_2$.

- If R goes first, they will win if and only if $x - (s_1 - s_2) < 0$, or $x < s_1 - s_2$. In other words, L wins if and only if $x \geq s_1 - s_2$.

By intersecting the cases, we can find the conditions for the outcome classes.

- Independent of who goes first, L wins exactly when both $-x < s_1 - s_2$ and $x \geq s_1 - s_2$. $s_1 - s_2 \geq 0$, so in order for that second condition to hold, x must also be non-negative. If $x = 0$, however, then $0 < s_1 - s_2 \leq 0$, which also cannot happen. Thus, we can better describe this case as $o(G) = \mathcal{L}$ iff $x > 0$ and $s_1 - s_2 \leq x$.

- Independent of who goes first, R wins exactly when both $-x \geq s_1 - s_2$ and $x < s_1 - s_2$. For similar rationale as above, $o(G) = \mathcal{R}$ iff $x < 0$ and $s_1 - s_2 \leq -x$.

- The first player wins exactly when $-x < s_1 - s_2$ and $x < s_1 - s_2$. Thus, $G \in \mathcal{N}$ iff $-(s_1 - s_2) < x < s_1 - s_2$.

- The previous player wins exactly when $-x \geq s_1 - s_2$ and $x \geq s_1 - s_2$. Thus, $G \in \mathcal{P}$ iff $s_1 - s_2 \leq x \leq -(s_1 - s_2)$. This last case can only happen when $x = 0$ and $s_1 = s_2$. (We save a proof of this for exercise 4.2.16 .)

Let's summarize this in one equation (with cases):

$$x \pm s_1 \pm s_2 \in \begin{cases} \mathcal{N}, & \text{if } |x| < s_1 - s_2 \\ \mathcal{L}, & \text{if } x > 0 \text{ and } s_1 - s_2 \leq x \\ \mathcal{R}, & \text{if } x < 0 \text{ and } s_1 - s_2 \leq -x \\ \mathcal{P}, & \text{if } x = 0 \text{ and } s_1 = s_2 \end{cases}$$

This is very similar to our one-switch case, using $s_1 - s_2$ in place of s and with some differences along the boundaries. E.g. using $<$ instead of $\leq$.

What about three switches? What are the outcome class possibilities for $G = x \pm s_1 \pm s_2 \pm s_3$? If L goes first, the value after three moves will be: $x + s_1 - s_2 + s_3$. If R goes first, the value after three moves will be: $x - s_1 + s_2 - s_3$. Since the first player also takes the last switch, this looks a lot like our one-switch case:

$$x \pm s_1 \pm s_2 \pm s_3 \in \begin{cases} \mathcal{N}, & \text{if } |x| \leq s_1 - s_2 + s_3 \\ \mathcal{L}, & \text{if } s_1 - s_2 + s_3 < x \\ \mathcal{R}, & \text{if } s_1 - s_2 + s_3 < -x \end{cases}$$

This pattern is going to continue where the outcome class of $x \pm s_1 \pm s_2 \pm s_3 \pm \cdots \pm x_n$ depends on three things:

- x
- $t = s_1 - s_2 + s_3 - s_4 + s_5 + \cdots + (-1)^{n+1} s_n$
- The parity of n.

For an even $n = 2k$, we have:

$$x \pm s_1 \pm s_2 \pm s_3 \pm s_4 \pm \cdots \pm s_{2k} \in \begin{cases} \mathcal{N}, & \text{if } |x| < t \\ \mathcal{L}, & \text{if } x > 0 \text{ and } t \le x \\ \mathcal{R}, & \text{if } x < 0 \text{ and } t \le -x \\ \mathcal{P}, & \text{if } x = 0 \text{ and } t = 0 \end{cases}$$

When $n = 2k + 1$ (n is odd), we get:

$$x \pm s_1 \pm s_2 \pm s_3 \pm s_4 \pm \cdots \pm s_{2k+1} \in \begin{cases} \mathcal{N}, & \text{if } |x| \le t \\ \mathcal{L}, & \text{if } t < x \\ \mathcal{R}, & \text{if } t < -x \\ \mathcal{P}, & \text{cannot happen} \end{cases}$$

Now let's put everything together. What is the outcome class of G?

$$G = x + a \uparrow + *k \pm s_1 \pm s_2 \pm \cdots \pm s_n$$

(where all variables are numbers and $s_1 \ge s_2 \ge s_3 \ge \cdots \ge s_n$)

We can combine our tactics above to solve this. Just as before, $t = s_1 - s_2 + s_3 - \cdots + (-1)^{n+1} s_n$. We can consider that after the first n moves, the value of the game will be depends on who goes first.

- If L goes first, then, after n turns, the value will be $x + t + a \uparrow + *k$.
- If R goes first, then, after n turns, the value will be $x - t + a \uparrow + *k$.

Since we've already learned how to determine the outcome class of numbers plus arrows plus nimbers, we can determine who can win that game to determine the overall outcome class.

For example, consider:

$$G = 2 + \Downarrow + *12 \pm 3 \pm 2$$

$t = 1$, so after two turns to resolve the switches, the game will either be $3 + \Downarrow + *12$ or $1 + \Downarrow + *12$. Both cases are in $\mathcal{L}$, so $G \in \mathcal{L}$.

For a more difficult example, consider:

$$G = -3 + \uparrow + *3 \pm 6 \pm 3$$

This time, $t = 3$, and $-x = t$, so we need to take a closer look at the two positions that could occur after the first two moves.

- If L goes first, then after two moves, L will be playing first on $\uparrow + *3 \in \mathcal{L}$.
- If R goes first, then after two moves, R will be playing first on $-6 + \uparrow + *3 \in \mathcal{R}$.

Whoever goes first wins, so $G \in \mathcal{N}$.

Recall that in these evaluations, we're not figuring out how to add G to another game. This is for deciding the outcome class *after* all addition has happened.

We have not covered all cases here. In fact, we can't cover all the cases in a way that is easy to calculate. Back in the 1980's, F. L. Morris [5] showed that it is PSPACE-hard to calculate the winnability of the sum of a bunch of shallow partisan game values. This means that the best-known algorithms to calculate the winner take an exponential amount of time in the worst cases.

The reason for this is that game sums can contain components more complicated than numbers, arrows, nimbers, and switches. These components don't need to be much more complicated, just things of the form $\left\{ \{ 5 \mid 4 \} \,\middle|\, 2 \right\}$. Adding bunches of these together combine to create expansive game trees. Currently there is no algorithm to solve them significantly better than just exploring the entire game tree.

Exercises for 4.2

⋆ **0)** What is the outcome class of $G = 5 + \Uparrow + *6$? (Answer 4.2.0 in Appendix)

1) What is the outcome class of $G = -3 + \Uparrow + *3$?

2) Use a proof by strong induction to show that for any $k \geq 2$, $\uparrow > *k$.

⋆ **3)** What is the outcome class of $G = -10 \pm 5$? (Answer 4.2.3 in Appendix)

4) What is the outcome class of $G = -5 \pm 10$?

⋆ **5)** What is the outcome class of $G = 3 \pm 3$? (Answer 4.2.5 in Appendix)

6) What is the outcome class of $G = 5 \pm 7 \pm 4$?

⋆ **7)** What is the outcome class of $G = \pm 6 \pm 6$? (Answer 4.2.7 in Appendix)

8) What is the outcome class of $G = -4 \pm 2 \pm 2$?

⋆ **9)** What is the outcome class of $G = -2 \pm 7 \pm 3$? (Answer 4.2.9 in Appendix)

10) What is the outcome class of $G = 6 \pm 8 \pm 2$?

⋆ **11)** What is the outcome class of $G = \pm 6 \pm 3 \pm 3$? (Answer 4.2.11 in Appendix)

12) What is the outcome class of $G = -5 \pm 3 \pm 3 \pm 3$?

⋆ **13)** What is the outcome class of $G = 7 \pm 6 \pm 5 \pm 4 \pm 3$? (Answer 4.2.13 in Appendix)

14) What is the outcome class of $G = -1 \pm 6 \pm 5 \pm 4 \pm 3 \pm 2 \pm 1$?

15) What is the outcome class of $G = -1 \pm 6 \pm 6 \pm 5 \pm 4 \pm 3 \pm 3$?

16) Prove the case that $G = x \pm s_1 \pm s_2 \in \mathcal{P}$ iff $x = 0$ and $s_1 - s_2 = 0$. Hint: use two direct proofs, one in each direction.

⋆ **17)** What is the outcome class of $G = 1 \pm 1 + \uparrow + *$? (Answer 4.2.17 in Appendix)

⋆ **18)** What is the outcome class of $G = \pm 4 \pm 4 + \downarrow$? (Answer 4.2.18 in Appendix)

19) What is the outcome class of $G = \pm 4 \pm 4 + \downarrow + *$?

20) What is the outcome class of $G = 4 \pm 4 + \downarrow$?

★ **21)** What is the outcome class of $G = 4 \pm 4 + \downarrow + *$? (Answer 4.2.21 in Appendix)

22) What is the outcome class of $G = -4 \pm 4 + \downarrow$?

23) What is the outcome class of $G = -4 \pm 4 + \downarrow + *$?

★ **24)** What is the outcome class of $G = 2 \pm 7 \pm 1 + \Uparrow + *$? (Answer 4.2.24 in Appendix)

25) What is the outcome class of $G = 5 \pm 10 \pm 4 + \Downarrow + *4$?

★ **26)** What is the outcome class of $G = \pm 4 \pm 3 + \downarrow + *8$? (Answer 4.2.26 in Appendix)

27) What is the outcome class of $G = 2 \pm 8 \pm 6 + \uparrow + *2$?

28) What is the outcome class of $G = 3 \pm 4 \pm 3 \pm 2 + *4$?

★ **29)** What is the outcome class of $G = \pm \uparrow$? (Answer 4.2.29 in Appendix)

30) What is the outcome class of $G = \pm \uparrow + *$?

4.3. Which Move Should we Make?

Our next question, "Which of our possible moves is part of a winning strategy?" is only relevant if we already answered the previous question positively: we know we can win, so which of our moves keeps us winning? This is a bit different than the last section, perhaps deceptively so. The reason is that when analyzing the outcome class of a position, we can assume that the *canonical moves* are available. For example, in the game $G = \frac{1}{2} + \downarrow$, we can assume that the $\frac{1}{2}$ part is actually the game $\left\{ 0 \,\middle|\, 1 \right\}$, meaning R has a move, on G, to $1 + \downarrow$. It might be, however, that G is actually $\left\{ 0 \,\middle|\, \frac{3}{4} \right\} + \downarrow$. This is still equivalent to $\frac{1}{2} + \downarrow$, but the actual options are different.

Deciding which move to make on G is a step that needs to happen *after* we've evaluated G. The calculations we make at this step no longer work if G is just a component of a larger game. In general, we should be following this process:

1. Break G into a sum of independent components: $G = G_1 + G_2 + \cdots + G_k$.
2. Evaluate each of those components: $G_1 = v_1, G_2 = v_2, \ldots, G_k = v_k$.
3. Add the values together, so $G = v_1 + v_2 + \cdots + v_k$.
4. Determine the outcome class of G.
5. If you can win, make a winning move.[4]

This section is about that last step. How do we decide which move to make?

When we're evaluating the components, it will make our evaluation easier if our values use only the four basic game types we've seen so far: numbers, nimbers, arrows, and switches. We already know how to add these together and determine the outcome class. For example, we know what to do with this sum:

$$G = (7 \pm 3 \pm 1 + *3 + \uparrow) + (-5 \pm 10 \pm 3 \pm 2 + *2 + \Downarrow)$$

$$\begin{aligned} G &= (7 \pm 3 \pm 1 + *3 + \uparrow) + (-5 \pm 10 \pm 3 \pm 2 + *2 + \Downarrow) \\ &= 7 - 5 \pm 10 \pm 3 \pm 3 \pm 2 \pm 1 + *3 + *2 + \uparrow + \Downarrow \\ &= 2 \pm 10 \pm 2 \pm 1 + * + \downarrow \in \mathcal{N} \end{aligned}$$

We already have an idea of where we should play on one of these basic positions. In our example G above, the first player will always want to figure out the component with the ± 10 value and play there. In general, switches are great to play on, nimbers are indifferent towards the "score" (but can certainly alter the winnability), arrows are bad (but better than numbers), and numbers are the worst components to play on (recall the Number Avoidance Theorem). This heuristic order of choices works when G has a basic value. If we don't have that luxury, we will have to get a bit more creative.

There are two different ways we could go about finding a winning move:

[4]If you can't win, then make a sneaky losing move that might trick your opponent.

1. We could analyze each of the options to determine their outcome class. When we find one that is a winning move, we choose that one.

2. Or, if we already know the value of the overall position, we might be able to see the difference in value for each of the moves. Then we can just subtract that difference from the current value and check whether we're still in a winning outcome class.

Let's test that second idea with an example, to double-check with the heuristic we gave above. Consider this position, G :

$$G = \left\{ 3\frac{1}{2} \,\middle|\, 1\frac{1}{2} \right\} + \{ * \mid 0 \} + \{ * \mid 0 \} + \left\{ -1 \,\middle|\, -\frac{1}{4} \right\} + \{ -1 \mid 3 \}$$

We know how to find the value of G:

$$\begin{aligned} G &= \left\{ 3\frac{1}{2} \,\middle|\, 1\frac{1}{2} \right\} + \{ * \mid 0 \} + \{ * \mid 0 \} + \left\{ -1 \,\middle|\, -\frac{1}{4} \right\} + \{ -1 \mid 3 \} \\ &= 2\frac{1}{2} \pm 1 + \downarrow + \downarrow - \frac{1}{2} + 0 \\ &= 2 \pm 1 + \Downarrow \\ &\in \mathcal{L} \end{aligned}$$

L must have a winning move. Let's look at each of the options for L and see how that changes the value.

$$\underbrace{\left\{ 3\frac{1}{2} \,\middle|\, 1\frac{1}{2} \right\}}_{a} + \underbrace{\{ * \mid 0 \}}_{b} + \underbrace{\{ * \mid 0 \}}_{c} + \underbrace{\left\{ -1 \,\middle|\, -\frac{1}{4} \right\}}_{d} + \underbrace{\{ -1 \mid 3 \}}_{e}$$

a , $\left\{ 3\frac{1}{2} \,\middle|\, 1\frac{1}{2} \right\}$: This component equals $2\frac{1}{2} \pm 1$. Let's consider the base of that ($2\frac{1}{2}$) to be an "average value", then L can increase that by 1 by playing here. The final value after moving here would be: $3 + \Downarrow$.

b , $\{ * \mid 0 \}$: This component equals $\downarrow$. Playing here replaces this with $*$, so the overall value changes by $* - \downarrow = * + \uparrow$. This is only infinitesimal, but it is still a change (and in kind of the right direction). The overall value after playing here would be: $2 \pm 1 + \downarrow + *$.

c : This is the same position as in part *b*.

d , $\left\{ -1 \,\middle|\, -\frac{1}{4} \right\}$: This component equals $-\frac{1}{2}$, so moving here changes the overall value by $-1 - (-\frac{1}{2}) = -\frac{1}{2}$. The final value after playing here would be: $1\frac{1}{2} \pm 1 + \Downarrow$.

e , $\{ -1 \mid 3 \}$: This component equals 0, so moving here changes the overall value by $-1 - 0 = -1$. The overall value after playing here would be: $1 \pm 1 + \Downarrow$.

Of all of these, the move after *a* is the largest, so playing in the switch is the best option[5].

As we mentioned before, we now have all the tools we need to play on positions that nicely evaluate into our four basic value types. Unfortunately, this doesn't help us when there's something that doesn't fall into these categories, e.g. $\left\{ 9 \,\middle|\, \{ 6 \mid 4 \} \right\}$.

We know how to deal with the right-hand side of this: $\left\{ 9 \,\middle|\, \{ 6 \mid 4 \} \right\} = \{ 9 \mid 5 \pm 1 \}$. Additionally, we can certainly determine the outcome class ($\mathcal{L}$) and figure out what to do if this is the entire game. If it's not, we don't have a good way to add this to the rest of the components, nor decide whether this is where we should be making our move.

Let's address this second part by introducing some new terms for studying these *hot games*. In particular, we will want:

- The *mean value* (or *fair value*) of a position. This will be the average value if two players were to optimally play on a sum of multiple copies of that position.

[5] We leave showing this for exercise 4.3.0 .

- The *temperature* of a position, which will be the "payoff" for a player to go first in that game.

With a switch, $G = a \pm b$, we already defined these parts, but with different names. The base of G, a, is also its mean value, and the heat of G, b, is also its temperature.

How can we define these more general terms for positions such as $\left\{ 9 \,\middle|\, \{ 6 \mid 4 \} \right\}$? We'll need to start by using stops, which we defined back in section 3.5. This time, we'll want to *adorn* the stops, by also indicating who took the last turn. For example, the left stop of ± 2, $LS(\pm 2)$, was 2. We'll update that by adding a subscript L to identify that L took the last turn. Thus, $LS(\pm 2) = 2_L$ and $RS(\pm 2) = -2_R$.

We want to identify the *confusion interval* for each of these hot games. The confusion interval of a game, G, is the set of numbers[6] confused with G. For ± 2, the confusion interval is $[-2, 2]$, which is the set of all numbers between -2 and 2, including both endpoints.

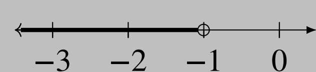
Math Diversion: Interval notation
Early in our study of mathematics we start to compare values using inequalities. We write the solutions to statements like

$$2x + 6 \leq 4$$

with their own inequalities:

$$x \leq -1.$$

This is visualized with the following number line.



We can also write this in *interval notation* as

$$x \in (-\infty, -1].$$

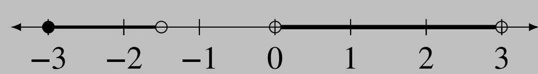
[6]The confusion interval strictly contains numbers, not all game values.

This says that x is in the set of all real numbers from $-\infty$ up to and including -1. Note the square bracket] at the right. This signifies that -1 is included in the set, while the parenthesis (or, *shudder*, the *open bracket*) on the left signifies that $-\infty$ is not in the set of solutions. We very rarely see ∞ or $-\infty$ included in a set of solutions since they are not real numbers, (though they do come up when considering *extended reals*, and we've seen something similar in the game value ω).

If we want to represent a disjoint collection of intervals, then we simply use the union operator.

$$|2x + 1| > 1$$
$$(2x + 1) > 1 \text{ or } (2x + 1) < -1$$
$$x > 0 \text{ or } x < -1$$
$$(0, \infty) \cup (-\infty, -1)$$

How would you write the solution represented by the following number line?



A: *This includes all values from* -3 *to* -1.5 *including* -3 *but not including* -1.5*, and all values between* 0 *and* 3 *not including either endpoint. We write this* $[-3, -1.5) \cup (0, 3)$.

When we use a square bracket then we say that side of the interval is *closed*, and a parenthesis signifies that side is *open*. This terminology comes from mathematical analysis where sets are considered open, closed, neither or both (*clopen*) depending on the metric.

Note that the interval $(0, 0)$ is empty, and $[0, 0] = \{0\}$ has only one element.

If $RS(G) = a_{P_1}$ and $LS(G) = b_{P_2}$, then our confusion interval is going to be between a and b, and may or may not include either end[7]. Each end of the

[7] Be careful here! It can be easy to think that the Left Stop is on the left-hand-side of the interval

interval is closed if the stop's adornment is the same as the player who went first, and open otherwise. So, for example, if $RS(G) = a_R$ and $LS(G) = b_R$, then the confusion interval of G is: $[a, b)$.

The mean value of any position G is contained between the ends, even if the confusion interval is open on one or both ends. If the confusion interval of G is (a, b) and $a = b$, then a is the mean value of G.

Above we said that we can use multiple copies of a game to find its mean value. To do this, we will find the stops of $n \times G$, then divide them by n. This can give us a more-narrow confusion interval, where the mean value still has to reside. More formally, $\forall n \in \mathbb{N} \setminus \{0\}$: the mean value of G, $m(G)$, must be between $RS(n \times G)/n$ and $LS(n \times G)/n$.

Let's look at $G = \pm 2$ as an example and check out the confusion intervals of $n \times G$ for different copies of n.

- For G, we already know that the confusion interval is $[-2, 2]$. Thus, $-2 \leq m(\pm 2) \leq 2$.
- For $2 \times G$, we need to consider optimal moves on $\pm 2 \pm 2 = \left\{ 2 \,\middle|\, -2 \right\} + \left\{ 2 \,\middle|\, -2 \right\}$. In this case, it's pretty simple, as each player will play once on each of the switches. Thus, $LS(2 \times G) = 0_R$ and $RS(2 \times G) = 0_L$, so the confusion interval of $2 \times G$ is $(0, 0)$. Thus, $\frac{0}{2} \leq m(\pm 2) \leq \frac{0}{2}$, so $m(\pm 2) = 0$.
- Let's take a look at $3 \times G$ just to see what happens. Here, we are finding the optimal moves on $G + G + G$. For L, the first move will be to $2 + G + G$, then R moves to $0 + G$, then L moves to 2, so $LS(3 \times G) = 2_L$. Similarly, $RS(3 \times G) = -2_R$, so we again have the confusion interval: $[-2, 2]$. This does help us a bit, however, because we divide each of those bounds by $n = 3$, so we would know that: $-\frac{2}{3} \leq m(G) \leq \frac{2}{3}$. Naturally, this doesn't matter, as we already found the correct value in the previous iteration.

We could go further with the above example. At every even mulitple of G, we would find that $m(G) = 0$, while the odds would tell us that $-\frac{2}{n} \leq m(G) \leq \frac{2}{n}$.

Let's try this with our other example, $G = \left\{ 9 \,\middle|\, \left\{ 6 \,\middle|\, 4 \right\} \right\}$. Here we'll see that the midpoint of the confusion interval is not always the mean value. Let's let $H = n \times G$ and test out different values of n:

(and vice versa), but in this case it's the other way around.

$n = 1$: $LS(H) = 9_L$, and $RS(H) = 6_L$, so the confusion interval is $(6, 9]$, and $6 \leq m(G) \leq 9$.

$n = 2$: $LS(H) = 15_l$. For $RS(H)$, after R moves to $\{ 6 \mid 4 \} + G$, it's better for L to play in G than the switch. Thus, $RS(H) = 13_R$, the confusion interval of H is $[13, 15]$, so $6 + \frac{1}{2} \leq m(G) \leq 7 + \frac{1}{2}$. Notice that the midpoint of this interval (7), is not the midpoint of the previous one ($7 + \frac{1}{2}$).

$n = 3$: As we reasoned before, it's better for L to play on G than the switch, so $LS(H) = 22_L$, while $RS(H) = 19_R$. Thus, the confusion interval of H is $[19, 22]$, and $6 + \frac{1}{3} \leq m(G) \leq 7 + \frac{1}{3}$. At this point, the midpoint of the interval when $n = 1$ isn't even an option any longer.

$n = 4$: $LS(H) = 29_R$, and $RS(H) = 29_L$, so the confusion interval of H is $(29, 29)$, and furthermore, $7 + \frac{1}{4} \leq m(G) \leq 7 + \frac{1}{4}$, meaning $m(G) = 7 + \frac{1}{4}$.

Let's do another example with very similar numbers: $G = \left\{ 7 \,\middle|\, \{ 6 \mid 4 \} \right\}$. In our analysis of multiples of G, let's note that if L has to decide whether to play on a single G component or $\{ 6 \mid 4 \}$, they will choose to avoid 4 and play on the 6 instead. Let's again consider $H = n \times G$ and test this for different values of n:

- $n = 1$: $LS(H) = 7_L$ and $RS(H) = 6_L$, so the confusion interval is $(6, 7]$ and $6 \leq m(G) \leq 7$.
- $n = 2$: $LS(H) = 13_L$ and $RS(H) = 12_L$, because L will play on the $\{ 6 \mid 4 \}$ component every time, as we mentioned above. Thus, the confusion interval is $(12, 13]$ and $6 \leq m(G) \leq 6 + \frac{1}{2}$.
- $n = 3$: $LS(H) = 19_L$ and $RS(H) = 18_L$, so the confusion interval is $(18, 19]$ and $6 \leq m(G) \leq 6 + \frac{1}{3}$.
- $n = 4$: $LS(H) = 25_L$ and $RS(H) = 24_L$, so the confusion interval is $(24, 25]$ and $6 \leq m(G) \leq 6 + \frac{1}{4}$.

There is definitely a pattern here. Due to the way L will react to R's moves every time, $LS(n \times G) = (7 + 6(n-1))_L = (1 + 6n)_L$ and $RS(n \times G) = 6n_L$. You may see that as n gets bigger, $\frac{LS(n \times G)}{n}$ tends towards 6. If you already know some Calculus, you know that we write this formally with limits: $\lim_{n\to\infty}\left(\frac{1+6n}{n}\right) = 6$. Even though our our iterative process above will never exactly isolate $m(G)$, by instead using the limit, we can see that $m(G) = 6$. (We can also see that the right stop will always be 6.)

Let's look at what happens if we try this with a game which is not hot. The position $G = \left\{ 6 \,\middle|\, \{ 6 \mid 4 \} \right\}$ is equal to $6 + +_2$. Let's see what happens with $H = n \times G$ when we find the confusion intervals. (As it turns out, we only have to try this with $n = 1$.)

When $n = 1$, $LS(H) = 6_L$ and $RS(H) = 6_L$. Thus, the confusion interval is $(6, 6]$ meaning $6 \le m(G) \le 6$. Our confusion interval is empty, however, as there are no numbers that are confused with G. Mean values are relegated to numbers, so saying that it is 6 doesn't fit because $G > 6$.

Now that we have some idea about the mean value (even if we can't find it in all cases), let's talk about the temperature. The *temperature* of a game G, is an indication of the importance of playing first in a game. In switches, this is equivalent to the heat. Temperature can be thought of as an extension of heat to all games[8]. In another characterization, the temperature, $t(G)$, is the minimum number of points a player would need to be awarded to give up going first on G. If they are offered less than $t(G)$, then they wouldn't be willing to pass to their opponent. If they are offered more than $t(G)$, they will always be happy taking the offer.

To calculate $t(G)$, we will need to use our notion of stops again, as well as the notion of *cooling*. $\check{G}_c$, or "G cooled by c degrees" is:

[8]It's not perfect, however, so it is in some ways an approximation for games more complicated than switches.

$$\breve{G}_c = \begin{cases} G & \text{, if } G \text{ is an integer} \\ \left\{ \breve{G}^{L}{}_c - c \;\middle|\; \breve{G}^{R}{}_c + c \right\} & \text{, if } \max\left(RS(\breve{G}^{L}{}_c)\right) - c \geq \min\left(LS(\breve{G}^{R}{}_c)\right) + c \\ \max\left(RS(\breve{G}^{L}{}_t)\right) - t & \text{, if } \exists t < c \text{ such that} \\ & \max\left(RS(\breve{G}^{L}{}_t)\right) - t = \min\left(LS(\breve{G}^{R}{}_t)\right) + t \end{cases}$$

, where c is a number $\in [-1, \infty)$. $\breve{G}^{L}{}_c - c$ means that we're recursively cooling each left option by c and then afterwards subtracting c from it. Note: the conditions in the definition above cover all the cases.

In other words, if the game is hot, then cooling it lowers the values on the left and raises the values on the right until the stops meet. The third case covers when cooling by c would otherwise lead to the stops passing each other. In that case, we have to back up and use t instead. That maximum cooling t is exactly the temperature, $t(G)$, and the value the sides converge to, $\max\left(RS(\breve{G}^{L}{}_t)\right) - t$ is exactly the mean value, $m(G)$.

Let's consider $G = \left\{ 3 \;\middle|\; \pm 2 \right\}$. Before we try cooling it, let's first find the confusion interval.

$$\begin{aligned} LS(G) &= 3 \checkmark \\ RS(G) &= LS(\pm 2) \\ &= LS(\left\{ 2 \;\middle|\; -2 \right\}) \\ &= 2 \checkmark \end{aligned}$$

Thus, the confusion interval is $[2, 3)$. Let's see what happens when we cool G by 10. We might try to start like this, but the second line is incorrect:

$$\begin{aligned}
\breve{G}_{10} &= \left\{\ 3 - 10 \ \middle|\ \breve{\pm 2}_{10} + 10\ \right\} \\
&= \left\{\ -7 \ \middle|\ \{\ 2 - 10 \mid -2 + 10\ \} + 10\ \right\} \text{ (incorrect)} \\
&= \left\{\ -7 \ \middle|\ \{\ -8 \mid 8\ \} + 10\ \right\} \\
&= \{\ -7 \mid 10\ \} \\
&= 0
\end{aligned}$$

The reason for this is that $\breve{\pm 2}_{10} \neq \{\ 2 - 10 \mid -2 + 10\ \}$, because there is a smaller cooling factor that sufficiently cools it until the stops are equivalent. Here's what happens if we ignore that second case in our definition and continue to the last case. (We'll use a $\rightarrow$ instead of $=$ because it's incorrect.)

$$\begin{aligned}
\breve{\pm 2}_{10} &\rightarrow \{\ 2 - 10 \mid -2 + 10\ \} \\
&= \{\ -8 \mid 8\ \}
\end{aligned}$$

Notice that the (max) stop of the left side is less than the (min) stop of the right side. This is an indication that we've gone too far! If you see that, you've cooled much and need to try a smaller u.

We can easily see what the smallest u would be here for this simple switch to make the stops meet:

$$\begin{aligned}
\left\{\ \breve{2}_u - u \ \middle|\ \breve{-2}_u + u\ \right\} &= \{\ 2 - u \mid -2 + u\ \} \\
&= \{\ 0 \mid 0\ \} \text{ for } u = 2
\end{aligned}$$

Let's try cooling G by 2:

$$\begin{aligned}
\breve{G}_2 &\rightarrow \left\{\ 3 - 2 \ \middle|\ \breve{\pm 2}_2 + 2\ \right\} \\
&= \left\{\ 1 \ \middle|\ \{\ 2 - 2 \mid -2 + 2\ \} + 2\ \right\} \\
&= \{\ 1 \mid * + 2\ \}
\end{aligned}$$

The stops aren't equal and the left is less than the right, so we've cooled this game too far. The distance between the two is 1, so we recommend cooling by 1/2 less:

$$\begin{aligned}\breve{G}_{3/2} \to & \left\{ 3 - {}^3\!/_2 \,\middle|\, \breve{\pm 2}_{3/2} + {}^3\!/_2 \right\} \\ = & \left\{ {}^3\!/_2 \,\middle|\, \{ 2 - {}^3\!/_2 \mid -2 + {}^3\!/_2 \} + {}^3\!/_2 \right\} \\ = & \left\{ {}^3\!/_2 \,\middle|\, \{ {}^1\!/_2 \mid -{}^1\!/_2 \} + {}^{3/2}/ \right\} \\ = & \{ {}^3\!/_2 \mid {}^3\!/_2 \pm {}^1\!/_2 \} = H\end{aligned}$$

$LS(H) = {}^3\!/_2 < 2 = RS(H)$, so we're still cooling too far (and $H \neq \breve{G}_{3/2}$). There's now a difference of ${}^1\!/_2$, so let's try cooling by ${}^1\!/_4$ less:

$$\begin{aligned}\breve{G}_{5/4} \to & \left\{ 3 - {}^5\!/_4 \,\middle|\, \breve{\pm 2}_{5/4} + {}^5\!/_4 \right\} \\ = & \left\{ {}^7\!/_4 \,\middle|\, \{ 2 - {}^5\!/_4 \mid -2 + {}^5\!/_4 \} + {}^3\!/_2 \right\} \\ = & \left\{ {}^7\!/_4 \,\middle|\, \{ {}^3\!/_4 \mid -sfrac34 \} + {}^{5/4}/ \right\} \\ = & \{ {}^7\!/_4 \mid {}^5\!/_4 \pm {}^3\!/_4 \} = H\end{aligned}$$

$LS(H) = {}^7\!/_4 < 2 = RS(H)$, so we're still cooling too far (and $H \neq \breve{G}_{5/4}$). There's now a difference of ${}^1\!/_4$. We could either try cutting that in half again, but that hasn't worked well, so we might rather try 1. Let's try 1:

$$\begin{aligned}\breve{G}_{1} \to & \left\{ 3 - 1 \,\middle|\, \breve{\pm 2}_{1} + 1 \right\} \\ = & \left\{ 2 \,\middle|\, \{ 2 - 1 \mid -2 + 1 \} + 1 \right\} \\ = & \left\{ 2 \,\middle|\, \{ 1 \mid -1 \} + 1 \right\} \\ = & \{ 2 \mid 1 \pm 1 \} = H\end{aligned}$$

$LS(H) = 2 = RS(H)$, and no smaller cooling amount will keep the stops equal, so we've cooled by the correct amount. $t(G) = 1$ and $m(G) = 2$.

Note that we mentioned possible values of t are in the range $[-1, \infty)$. There are some cases here:

- Negative-temperature games are *cold*. Numbers are cold games. $\breve{5}_{-1} = 5$, so -1 is the smallest value where the stops meet and are equal to the single-valued confusion interval. Similarly, $\breve{3/2}_{-1/2} = \left\{ 1 + \frac{1}{2} \,\middle|\, 2 - \frac{1}{2} \right\} = \{ 3/2 \mid 3/2 \}$. This is the lowest t that works, so $t(3/2) = 1/2$.

- Games with zero temperature are *tepid*. These are any games where the stops are already equal and include nimbers: $\breve{*}_0 = *$ and other infinitesimals.

How should we best find the correct t? With $G = \{ 3 \mid \pm 2 \}$, we just tried things out until they worked. As mentioned earlier, possible temperature values go from $[-1, \infty)$. In the above example, after we saw that $t(G) < 10$ because 10 supercooled it, we then knew $t(g) \in [-1, 10)$. Once we have two endpoints, we can perform a *binary search*, which means we take the midpoint of our two endpoints, test that, and if we haven't finished, we can try again.

Sometimes, however, we can be a little smarter about our choices. Since $LS(G) > RS(G)$, we know that we're going to need positive cooling, so our interval to search is actually $(0, 10)$. Furthermore, if cool a game by an additional c, each stop can move inward by at most c. Thus, if the difference between stops is at k, the fastest the stops can converge is by cooling by $k/2$ more. Since the initial difference in stops is 1, we need to cool by at least $1/2$, so the actual interval is $(1/2, 10)$.

We could use this tactic to start with a bottom-up search. If we started by cooling by $1/2$, as in one of the examples above, the space between the stops there is $1/2$, so we'd have to cool by at least $1/4$. However, if you notice that the left stop came in from 3 to $5/2$ and the right stop didn't change, you might expect this pattern to continue and choose to cool by another $1/2$. And, if you did overshoot with this, you'd have a very short interval to search. Of course, in this case, it turns out to be the correct amount to cool by.

Temperature is a good heuristic to use to choose which component to play on. It works when all components are switches, numbers, and nimbers. Choosing to play on the game with the highest temperature is a strategy known as *hotstrat*. It is possible, however, for *hotstrat* to pick the wrong component to play on. For example, consider the game $\pm 1 + \pm 1 + \mathord{+}_{100}$. Here, if L plays on one of the components with temperature 1, then R will respond on $\mathord{+}_{100}$, yielding $\pm 1 + 1 + \{\, 0 \mid 100 \,\}$. Then L must respond on the hotter switch, leaving $\pm 1 + 1$, from which R can move to 0 and win. However, if L had instead started by playing in the tepid infinitesimal component, they would leave $\pm 1 + \pm 1 = 0$, and thus win the game.

Exercises for 4.3

0) Prove that in the first example in the chapter, $G = \{\, 3^1\!/\!_2 \mid 1^1\!/\!_2 \,\} + \{\, * \mid 0 \,\} + \{\, * \mid 0 \,\} + \{\, -1 \mid -^1\!/\!_4 \,\} + \{\, -1 \mid 3 \,\}$, the best move for L is to play on the switch component $\left(\{\, 3^1\!/\!_2 \mid 1^1\!/\!_2 \,\}\right)$. Write the proof by comparing that option to the other Left options. (Note: the solution to this is longer than you might expect.)

⋆ **1)** Consider $G = \left\{\, \{\, 4 \mid 2 \,\} \,\middle|\, -2 \,\right\}$. Find the left and right stops and the confusion intervals of $G, 2 \times G$, $3 \times G$, and $4 \times G$. Is this information enough to find the mean value of G, $m(G)$? If so, find $m(G)$. Additionally, find the temperature, $t(G)$. (You do not need to show calculations of $\breve{G}_t$ that reach the incorrect temperature.) (Answer 4.3.1 in Appendix)

2) Consider $G = \left\{\, 10 \,\middle|\, \{\, 3 \mid 2 \,\} \,\right\}$. Find the left and right stops and the confusion intervals of $G, 2 \times G$, $3 \times G$, and $4 \times G$. Is this information enough to find the mean value of G, $m(G)$? If so, find $m(G)$. Additionally, find the temperature, $t(G)$. (You do not need to show calculations of $\breve{G}_t$ that reach the incorrect temperature.)

⋆ **3)** Consider $G = \left\{\, 100 \,\middle|\, \{\, -10 \mid -20 \,\} \,\right\}$. Find the left and right stops and the confusion intervals of $G, 2 \times G$, $3 \times G$, and $4 \times G$. Is this information enough to find the mean value of G, $m(G)$? If so, find $m(G)$. Additionally, find the temperature, $t(G)$. (You do not need to show calculations of $\breve{G}_t$ that reach the incorrect temperature.) (Answer 4.3.3 in Appendix)

4) Consider $G = \left\{ \{ -2 \mid -10 \} \,\middle|\, -20 \right\}$. Find the left and right stops and the confusion intervals of $G, 2 \times G$, $3 \times G$, and $4 \times G$. Is this information enough to find the mean value of G, $m(G)$? If so, find $m(G)$. Additionally, find the temperature, $t(G)$. (You do not need to show calculations of $\breve{G}_t$ that reach the incorrect temperature.)

⋆ **5)** Consider $G = \left\{ 2 \,\middle|\, \{ 0 \mid -2 \} \right\}$. Find the left and right stops and the confusion intervals of $G, 2 \times G$, $3 \times G$, and $4 \times G$. Is this information enough to find the mean value of G, $m(G)$? If so, find $m(G)$. Additionally, find the temperature, $t(G)$. (You do not need to show calculations of $\breve{G}_t$ that reach the incorrect temperature.) (Answer 4.3.5 in Appendix)

6) Consider $G = \left\{ \{ 1 \mid 0 \} \,\middle|\, -5 \right\}$. Find the left and right stops and the confusion intervals of $G, 2 \times G$, $3 \times G$, and $4 \times G$. Is this information enough to find the mean value of G, $m(G)$? If so, find $m(G)$. Additionally, find the temperature, $t(G)$. (You do not need to show calculations of $\breve{G}_t$ that reach the incorrect temperature.)

⋆ **7)** Consider $G = \left\{ \{ 7 \mid 0 \} \,\middle|\, -5 \right\}$. Find the left and right stops and the confusion intervals of $G, 2 \times G$, $3 \times G$, and $4 \times G$. Determine $m(G)$ from the limit of the stops of $n \times G$, then find $t(G)$. (Answer 4.3.7 in Appendix)

8) Consider $G = \left\{ 100 \,\middle|\, \{ 75 \mid 0 \} \right\}$. Find the left and right stops and the confusion intervals of $G, 2 \times G$, $3 \times G$, and $4 \times G$. Determine $m(G)$ from the limit of the stops of $n \times G$, then find $t(G)$.

⋆ **9)** Consider $G = \left\{ \{ -10 \mid -20 \} \,\middle|\, -25 \right\}$. Find the left and right stops and the confusion intervals of $G, 2 \times G$, $3 \times G$, and $4 \times G$. Determine $m(G)$ from the limit of the stops of $n \times G$, then find $t(G)$. (Answer 4.3.9 in Appendix)

10) Consider $G = \left\{ \{ 19 \mid 1 \} \,\middle|\, -1 \right\}$. Find the left and right stops and the confusion intervals of $G, 2 \times G$, $3 \times G$, and $4 \times G$. Determine $m(G)$ from the limit of the stops of $n \times G$, then find $t(G)$.

⋆ **11)** Consider $G = \left\{ \{ 10 \mid 9 \} \,\middle|\, \{ 2 \mid 0 \} \right\}$. Use the tactics we learned to find $m(G)$ and the temperature of $t(G)$. Make sure you explain the optimal strategies each player uses while describing the stops! (Answer 4.3.11 in Appendix)

12) Consider $G = \left\{ \{ 10 \mid 0 \} \,\middle|\, \{ -1 \mid -9 \} \right\}$. Use the tactics we learned to find $m(G)$ and the temperature of $t(G)$. Make sure you explain the optimal strategies each player uses while describing the stops!

⋆ **13)** Consider $G = \left\{ \{ 5 \mid 3 \} \,\middle|\, \{ 2 \mid 1 \} \right\}$. Use the tactics we learned to find $m(G)$ and the temperature of $t(G)$. Make sure you explain the optimal strategies each player uses while describing the stops! (Answer 4.3.13 in Appendix)

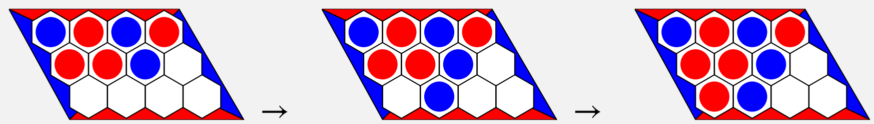
14) Consider $G = \left\{ \{ 100 \mid -1 \} \,\middle|\, \{ -2 \mid -3 \} \right\}$. Use the tactics we learned to find $m(G)$ and the temperature of $t(G)$. Make sure you explain the optimal strategies each player uses while describing the stops!

4.4. More strategy stealing

Many combinatorial games, both impartial and partisan, involve pieces that are placed and which then remain unmoved throughout the duration of play. Such games are called *placement games*. COL is one example that we've seen, but so are CHOMP, NODE KAYLES, and many others since moves can be imagined as placing blocking pieces on the board. CLOBBER and KONANE are not placement games since pieces are moved around the board. In this section we introduce another placement game with some interesting mathematical properties.

HEX

HEX is a partisan placement game played on a parallelogram-shaped board of hexagonal spaces. The four edges of the board are colored, blue on the left and right, and red on the top and bottom. Players take turns adding a single piece of their color, and the first to complete an unbroken line of monochromatic pieces of their color connecting their two sides of the board wins.



Left places a piece, then Right blocks for the win.

HEX is played in tournaments and has been studied pretty extensively. It's not an easy game to solve, which makes it more fun to play than, say, NIM, which we all know how to win at this point in the book. A number of HEX variants have sprung up, including FOLLOW THE LEADER where one player must, if possible, place their piece next to the opponent's previously placed piece.

One thing you have probably noticed about HEX is that game play ends not when the board is full, but once a winning condition has been met. CGT usually assumes that play will continue until neither player has a move remaining, but we've established an actual condition under which a player has won even if playable spaces remain on the board. This is familiar to every reader who has player TIC-TAC-TOE/NAUGHTS AND CROSSES. There are generally two ways to handle this: Either the board becomes unplayable by both players immediately following a winning move, or the winner may continue to play on the board and the loser may not. You can see how a win on a HEX board under the latter convention can have a big effect on game sums, thus making the entire board itself relatively hot. HEX is also played under scoring game rules, wherein winner actually receive points based on the number of remaining empty spaces, but scoring game analysis is beyond the scope of this book.

Since HEX is relatively unique among the universe of combinatorial games in that a winner can be determined before all available spaces are played, it's natural to wonder whether a game always ends with a winner in this way. It turns out that it will, and the proof is simpler than you might expect.

Theorem 4.4.1. *Every completed game of* HEX *has an unbroken monochromatic path between the sides of the same color.*

Proof. We will prove this directly. Call the side of a hexagon an edge, and refer to the corners as ul, ur, ll, lr for upper left, upper right, lower left, and lower right, respectively. Consider a filled HEX board, and note that every edge has either red or blue on either side if we include the colored sides of the board itself. Draw an unbroken path P along edges beginning in ll in the following way: proceed from vertex u to vertex v if the edge uv has blue to its left and red to its right, and continue until reaching ul, ur, ll, or lr. Note that P will always end at one of the four corners of the board, since stopping in the middle of the board would imply that there is an edge with blue to the left and red to the right, but such a configuration is impossible since pieces are hexagonal and P may proceed around any given piece if necessary.

Now note that if P ends at ul then there is an unbroken red path of pieces between the top and bottom sides of the board, and hence R is the winner. Similarly, if P ends in lr then L is the winner. In the exercises you will determine the winner in the other two cases. □

As we noted above, HEX is quite hard to analyze. However, we *can* determine the winner on an empty board of dimension $n \times n$ in a familiar way. Recall in Section 0.8 that any empty rectangular board in CHOMPis in $\mathcal{N}$. We proved this using a strategy stealing argument. We can do the same with HEX, and the process can be applied to some other placement games, as well. We begin with a lemma.

Lemma 4.4.2. *If a player has a winning option on a* HEX *position H, then they also have a winning move on H', the position H with any extra piece of the player's color.*

Proof. Assume without loss of generality that L has a winning move in H, and that H' is H with an extra blue piece at position x. If L's winning move on H is to play on x, moving H to H', then they can play to another place and the argument follows inductively. Similarly, if L's winning move on H is to play on a different position y, then y is also free in H' and L can play there on H', as well. Again, an inductive argument carries us to the fact that H' is also a winning position for L. □

Now we proceed with our strategy stealing argument with a proof by contradiction, much like we did for the proof of Theorem 0.8.1.

Theorem 4.4.3. *Any empty* HEX *board is in* $\mathcal{N}$.

Proof. Assume that a given empty board B of dimensions $n \times n$ is in $\mathcal{P}$, so the second player has a strategy S to win, and without any loss of generality, assume that L is the first player. L can make an arbitrary move to position x_0, and then follow the strategy S as if R was the first to move. If at any point S calls for the position x_0 to be played, then L can instead play to any other open position x_1, and continue this process. By Lemma 4.4.2 the existence of an extra blue piece at any position x_i does not negatively affect L's winning strategy. The winning strategy S has been co-opted by L and hence the game is in $\mathcal{N}$, contradicting our assumption that such strategy for the second player exists. □

Interestingly, this argument was first introduced by John Nash but not published until a paper on TIC-TAC-TOE in 1963 [**?**].

Exercises for 4.4

0) Complete the proof of Theorem 4.4.1.

⋆ **1)** Determine the winner on an empty HEX board of height 2 and width n. (Answer 4.4.1 in Appendix)

2) Determine the winner on an empty HEX board of height 3 and width n.

3) Use a strategy stealing argument to prove that the first player cannot lose in TIC-TAC-TOE.

⋆ **4)** Using the above result, show that TIC-TAC-TOE always ends in a tie when played perfectly by both players. (Answer 4.4.4 in Appendix)

5) Is an empty HEX board of dimensions $n \times n$ under the Follow The Leader convention (FTL) always in $\mathcal{N}$? Prove or disprove.

5. Non-combinatorial games

While we have been primarily focusing on combinatorial games, any introduction to the mathematics of games should include a discussion on other types of games. We have all played, and sometimes enjoyed, games that use cards, dice, or spinners. These are non-combinatorial games since they contain hidden information or depend on probability. But we can still get deeply into the mathematics of these games while remaining true to the focus of this text.

5.1. Card games

A standard deck of 52 cards contains 13 ranks: $2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A$, each appearing on cards of four suits: $\heartsuit$hearts, $\clubsuit$ clubs, $\diamondsuit$diamonds, and $\spadesuit$ spades.

You should note here that there are 1326 pairs of distinct cards in the deck: $2\spadesuit 2\heartsuit$, $2\spadesuit 3\heartsuit$, $2\spadesuit 4\heartsuit$, etc.

Now spend a little time thinking about how many sets there are of three distinct cards in the deck. You should find 22 100, and at the same time realize that this is a lot of work. As you can see, these numbers increase pretty rapidly as n, the number in each set, increases. In fact, if $n = 5$ then there are 2 598 960 possible hands.

These are examples of *combinatorial problems*. The term *combinatorics* is used for counting things, often large sets of things, using mathematical methods.

This increase we see as n increases is called *combinatorial explosion*. This is when one factor in a situation increases and another aspect increases far more rapidly.

We would like to be able to count the sizes of these sets easily. In order to do that we need to build up a bit more of a mathematical foundation of counting outside of the theory of games.

Math Diversion: Counting with repetition

Say that we want to generate all possible passwords of length exactly eight, where each character follows the following restrictions. How many passwords are possible in each case?

Lowercase letters or digits (e.g. $ngh4k628$)

> **A:** $(26 + 10)^8 \approx 2.8 \times 10^{12}$

Lowercase or uppercase letters, digits (e.g. $nGH4k628$)

> **A:** $(26 + 26 + 10)^8 \approx 2.2 \times 10^{14}$

No longer limited to 8 characters, how many passwords are possible using exactly three words from the Oxford English Dictionary, which currently contains 171 476 words. Repetition of words is permitted. (e.g. *glamorous-truculent-spell*)

> **A:** $(171\,476)^3 \approx 5 \times 10^{15}$, *and far easier to remember*

Note how including uppercase letters increases the possible number of passwords by a factor of almost 100.

These are examples of combinatorial problems where each option can be repeated. Once a character is used in a password there is no reason it can't be used again. We can increase the *entropy* of the password (how unpredictable it is) by allowing variable length passwords. Let's repeat the above questions but with variable passwords lengths.

How many passwords are possible that use lowercase letters or digits and have length 3 up to 8?

> **A:** $36^3 + 36^4 + \cdots + 36^8 \approx 2.9 \times 10^{12}$

What about lowercase or uppercase letters or digits, with length 3 up to 8?

A.: $62^3 + \cdots + 62^8 \approx 2.2 \times 10^{14}$

Notice that the entropy is not increased by that much over the restriction of fixed length passwords. Also, the clever Calculus student will notice a partial sum.

If a password has length n_0 to n_1 and is generated from a pool of x-many distinct characters, then write out how many passwords are possible using a series, and find a closed-form solution.

A.: $\sum_{k=n_0}^{n_1} x^k$. *Use the formula* $S_n = a\frac{r^n-1}{r-1}$ *with* $a = x^{n_0}$ *and* $n = (n_1-n_0+1)$ *to get* $x^{n_0}\left(\frac{x^{n_1-n_0+1}-1}{x-1}\right)$.

Math Diversion: Counting permutations

Sometimes we cannot reuse values. The usual examples are a relay team or a class seating chart, so let's use something else.

Say that you've recently angered the local clowning guild by dodging a squirting flower on one's lapel leading to the stream hitting a local police officer. The clowns have promised to exact revenge. All twelve members of the guild pull up to you in a tiny car, and three of them plan to climb out, one at a time, to attack you with pies. How many ways can they do this?

A.: $(12)(11)(10) = 1320$

What if all twelve plan to attack?

A.: $12! = 479\,001\,600$

There is a great deal more we can do with permutations but this is enough for our study of games. Now it's time to move to counting without repetition, but where the order no longer matters.

Math Diversion: The binomial coefficient and counting combinations

Combinations, collections of elements forming an unordered set, are probably the most relevant to our study of games. Let's stick with our clown guild.

The clowns want to bake pies. They do not care how good they taste. There are seven different fillings they can use: pumpkin, whipped cream, shredded coconut, blueberry, rhubarb, peach, and chicken. How many pies are possible if each pie has one, two, or seven ingredients?

A.: *There are 7 pies with one ingredient, 21 pies with exactly two, and only one pie with all seven ingredients.*

To find how many are possible with exactly 4 ingredients is difficult using *brute force*, i.e. just trying to list them all. Instead, consider the act of choosing ingredients. If you need 4 and it matters in what order you select them, then there are $(7)(6)(5)(4) = 840$ ways to do this. Note that this is $\frac{7!}{(7-4)!}$. However, in this problem we just want to select 4 fillings and set them on the counter to be added to the recipe; the way they're lined up on the counter isn't important. So, for example, pumpin-coconut-peach-chicken is equivalent to pumpkin-coconut-chicken-peach. Therefore we need to divide this number by the number of ways to arrange the four we've chosen.

How many 4-filling pies are possible?

A.: $\frac{7!}{(7-4)!4!} = 35$

In general, if the clowns have n-many fillings from which to choose and they want to choose k-many for a pie, how many ways can they do so? This is denoted $\binom{n}{k}$ and called the *binomial coefficient*.

A.: $\binom{n}{k} = \frac{n!}{(n-k)!k!}$

Instead, the clowns start with a pie crust and make as many distinct pies as possible: pumpkin, pumpkin-whipped cream, one with all seven ingredients, etc. How many pies do they make with at least one filling?

A.: $2^7 - 1 = 127$ *pies*

Now we return to cards. A standard poker hand is five cards. For each of the following, determine how many possible ways there are to get the given hand. Assume A is the highest rank and can't be treated as below 2.

For example, a *flush* is all five cards of the same suit. One example of a flush is $2,7,J,4,9\heartsuit$ and another is $3,7,J,4,9\heartsuit$. In order to get a flush, we must have a suit $\binom{4}{1}$ and then five ranks $\binom{13}{5}$. So the total number of flushes is $\binom{4}{1}\binom{13}{5}$.

As another example, two pairs could be $2\diamondsuit 2\spadesuit 4\clubsuit 4\diamondsuit J\spadesuit$. To count the number of distinct hands with two pairs that does not contain three-of-a-kind, we first pick two ranks, then two suits in each rank, then finally a fifth card from the remainder of the deck that doesn't match either of the chosen ranks. So the answer is $\binom{13}{2}\binom{4}{2}\binom{4}{2}\binom{44}{1}$.

For the following hands, determine how many distinct combinations are possible.

- A straight (five ranks in a row regardless of suit, e.g. $5\diamondsuit 6\spadesuit 7\clubsuit 8\diamondsuit 9\spadesuit$)

- A straight flush (a straight but all in the same suit)

- A full house (two cards of one rank, three of another)

- Four of a kind (four of one rank, the fifth card can be anything else)

- A single pair that is not part of three of a kind, nor a full house

A: *The lowest card in a straight can be* $2-10$*, so there are nine such choices. Note that no matter how the cards are dealt there is only one way to arrange them if they happen to form a straight. Each can be any suit. So we get* $(9)4^5 = 9216$ *possible straights.*
There are only $\binom{4}{1}9 = 36$ *straight flushes.*
For each full house, pick a triple, then a pair. The triple can be any three suits, and the pair any two. There are $(13)(12)\binom{4}{3}\binom{4}{2} = 3744$ *full houses possible.*
For four of a kind pick a rank, then a fifth card. $(13)(48) = 624$.
Finally, to count the number of ways to get a single pair, pick a rank, then two suits, then three different ranks, and finally three suits. $13\binom{4}{2}\binom{12}{3}4^3 = 1\,098\,240$.

There is another useful tool for counting items in the union of sets called the *Inclusion/Exclusion Principle*.

Math Diversion: Inclusion/Exclusion

Let A and B be sets with some overlap, e.g. A represents the set of Mathematics majors and B the set of Computer Science minors at a university. We want to send invitations to all of them for a party - likely the greatest party ever. Knowing both $|A|$ and $|B|$ is insufficient to determine how many invitations we need.

If we order $|A| + |B|$ many invitations, then we've potentially wasted money ordering invitations for some people twice, namely those in the set $A \cap B$. So we will need to subtract off those extra invitations from the total count, leaving us with $|A \cup B| = |A| + |B| - |A \cap B|$. This is the *Inclusion/Exclusion Principle*.

Now say that we want to invite another group; C is the set of students who tutor mathematics. How do we use Inclusion/Exclusion to determine the number of invitations to order?

A: *This is the same as finding* $|A \cup B \cup C|$, *which is equal to* $|A|+|B|+|C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$.

Note that we needed to not only subtract the pairwise intersections, but then add the three-way intersection back on. That's because we added in $|A \cap B \cap C|$ three times, then subtracted it three times, and hence had to add it back one more time.

Now say that there are 20 Mathematics majors, 30 Computer Science minors, and 9 tutors. 5 Mathematics majors are tutors, 3 CS minors are tutors, and 2 students are all three. In total there are 50 students. How many students are Mathematics majors and CS minors but not tutors?

A: *By Inclusion/Exclusion we have* $50 = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$. *So* $40 = 20+30+9-|A \cap B|-3-5+2 \Rightarrow 50 = 53 - |A \cap B| \Rightarrow |A \cap B| = 3$

Exercises for 5.1

⋆ **0)** How many 6 character license plates are possible in which the first three characters are letters and the last three are digits? (Answer 5.1.0 in Appendix)

1) How many 6 or 7 character license plates are possible in which the first three characters are letters and the remaining characters are digits?

2) How many 6 or 7 character license plates are possible in which the first three characters are letters, no letters are repeated, and the remaining characters are digits?

⋆ **3)** Which is a stronger password? Justify.

1. 8 to 10 characters, including uppercase letters, lowercase letters, digits, and 32 other symbols, or

2. Any three to four 4 letter words, of which there are 3996 in the English language

(Answer 5.1.3 in Appendix)

4) How many 4 letter English words must be allowed for a password to be at least as secure as the stronger password above?

For the following questions (5.1.5 through 5.1.8) instead of being dealt 5 cards from a standard deck you are dealt 7 cards.

⋆ **5)** How many distinct flushes are possible? (Answer 5.1.5 in Appendix)

6) How many distinct straights are possible?

⋆ **7)** How many ways could you get three of one rank and four of another? (Answer 5.1.7 in Appendix)

8) How many ways could you get three pairs with the remaining card of a different rank than all the others?

⋆ **9)** Use the Inclusion/Exclusion Principle to determine how many positive integers less than or equal to 50 are a multiple of 2, 3, or 5. (Answer 5.1.9 in Appendix)

10) Determine a formula for the value $|A \cup B \cup C \cup D|$ using the sizes of the respective sets and their intersections. Justify your answer.

11) Using Inclusion/Exclusion, find the number of possible passwords composed of lowercase letters, uppercase letters, digits, and the five special characters in $\{., !@\&\}$ with the following restrictions.

1. The length must be $8 - 10$ characters

2. It must contain at least one lowercase letter, at least one uppercase letter, at least one digit, and at least one special character

5.2. Dice and role-playing games

Next we move onto dice; in particular, role-playing applications. In a number of role-playing game scenarios a player is tasked with rolling a handful of dice which they hope will sum to a high enough value to pass the test, defeat the monster, or survive the potentially fatal blow. These dice often have varying shapes. Assuming the reader knows nothing about dice (singular *die*), we will proceed with the following assumptions:

- an n-faced die is called a " dn "
- A dn has the same probability of landing on any one face as another
- Each face of a dn has a unique number, and these are $1, 2, \ldots, n$

So, for example, a d4 has 4 faces labeled $1, 2, 3, 4$, and each face has the same probability of turning up on a single roll. As a side note, most polyhedral dice that a player encounters are *regular polyhedra*, which means that the faces are identical regular polygons with equal angles and side lengths, and the same number of faces meet at each vertex. There is no regular polyhedron with exactly 3, 7, or 13 sides, but a curious collector can easily purchase a d3 , d7 , and a d13 . These dice don't necessarily have regular polygonal faces, but they are still *fair* in the sense that the probabilities are the same for each face. And, of course, a d2 is simply a coin.

Math Diversion: Discrete probability
The *probability* of a discrete event A occurring is equal to the ratio of the number of ways that it can occur times their weights to the number of total possible outcomes times their weights. Since we are considering unweighted dice (all faces have the same probability of coming up on a single roll) this means that we need only worry about unweighted probabilities:

$$P(A) = \frac{\text{outcomes that match } A}{\text{all possible outcomes}}$$

Note that a probability cannot ever be less than zero (we are only counting non-negative quantities) not greater than one (the number of events that match A is never greater than the number of possible outcomes). For example, what is the probability that a roll of a d12 results in an odd value greater than 4?

A.: *There are four outcomes that match this criterion:* $5, 7, 9, 11$. *Since there are twelve total faces that occur with equal probability, we get* $P(odd > 4) = \frac{4}{12} = \frac{1}{3}$.

Events are *independent* if the occurence of one does not have any effect on the occurrence of another. Otherwise the events are *dependent*. Dice rolling tends to fall into the former category since each time a die is rolled the probabilities are the same no matter what has occurred up to that point. Cards, for example, are often dependent since the probability of being dealt a Queen is different whether or not you already have one in your hand.

We can combine discrete probabilities in a number of ways. While a complete address of probability is outside the scope of this text, it's worth noting the basic `AND` and `OR` operations on events. The probability of independent events A and B both occurring is

$$P(A \cap B) = P(A) \cdot P(B)$$

while the probability of A or B is

$$P(A \cup B) = P(A) + P(B)$$

Now consider that you have a very generous friend willing to gamble with you using a d6 in the following way.

- If it lands on a value greater than 4 then she pays you \$5
- If it lands on 4 then she pays you \$1
- Otherwise, you pay her \$2

While a few plays of this game could result in very unexpected outcomes, we will consider what happens if you play the game with her many, many times. This way the *empirical* outcomes will very likely be close to the *theoretical* outcomes. In probability theory this is called the *Law of Large Numbers*.

On a single roll your *expected winnings* (also called the *expectation*) equals the sum of each outcome multiplied by the winnings in that case. In general,

$$Exp = P(a_1)W(a_1) + P(a_2)W(a_2) + \cdots + P(a_n)W(a_n)$$

where $\{a_1, a_2, \ldots, a_n\}$ is the set of possible outcomes, and $W(A)$ represents your winnings under outcome A. Note that the expectation is not what you expect to win in a single roll of the dice, but instead what you expect to win *on average* each roll over many plays of the game.

In your game, your expectation is $P(>4)\$5 + P(4)\$1 + P(<4)(-\$2) = \frac{1}{3}\$5 + \frac{1}{6}\$1 - \frac{1}{2}\$2$, which is about \$0.83. Therefore, if you play this game with your friend 1000 times then you can expect to come away with about \$83.

If your friend wants to change the rules to be more fair (and she's your friend so you oblige) by raising the amount you pay on a roll of 1, 2, or 3, then what should you change it to?

A.: *We want* $Exp = \frac{1}{3}\$5 + \frac{1}{6}\$1 - \frac{1}{2}\$x = 0$. *Solving for x yields approximately* \$3.67. *After* 1000 *plays of the game you both expect to come away with* \$0 *and a better relationship.*

In some role-playing games a player has a choice of which item to use in a given scenario. Let's assume that your character walks into a shop with a broadsword under her cloak. Your character is not very good at lying but is wearing an Amulet of Coercion which helps her out. She also has, well, a broadsword and shoulders wide enough to barely fit through the door of the shop, so she's quite intimidating. You need to roll at least 12 on an attempt to either bluff or intimidate. With the amulet your bluff, "It's just a large, oddly shaped wallet," gives you a d8 , 3 d4 , and a +2 modifier. To intimidate you roll 3 d6 . To determine which is the better option (at least mathematically), we will calculate the expectation of each die roll.

The expectation on a single d4 is $P(1)1 + P(2)2 + P(3)3 + P(4)4$. Since each probability is $(\frac{1}{4})$ this equals $\frac{1}{4}(1 + 2 + 3 + 4)$ or 2.5, just the mean of the values on the faces. In general the expectation from a single dn is

$$P(1)1 + P(2)2 + \cdots + P(n)n = \frac{1}{n}\sum_{i=1}^{n} i$$

If you've taken Calculus then you will recall that $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$, so we get that

$$Exp(\mathrm{d}n) = \frac{n(n+1)}{2n} = \frac{n+1}{2}$$

Therefore, we see that $Exp(\mathrm{d}4) = 2.5$, $Exp(\mathrm{d}6) = 3.5$, $Exp(\mathrm{d}8) = 4.5$, etc.

So, in our scenario, your expected rolls are:

$$\begin{aligned} Exp(\text{bluff}) &= 4.5 + 3(2.5) + 2 = 14 \\ Exp(\text{intimidate}) &= 3(3.5) = 10.5 \end{aligned}$$

Your expected outcome is much better with the bluff, especially with your amulet. Which is good, because there just happened to be a heavily-armed garrison just outside the door.

6. Game Properties

Now that we have learned about impartial and partisan games, and the values that these games can achieve, it's time to delve into some interesting properties of games. John Conway, when conceiving of the concept of a *game* as a new kind of number, established that they should act like numbers as much as possible. So, as we have seen, any two games that are rational numbers can be added together like rational numbers, adhere to $<, >, =$ like rational numbers, etc. But because the set of all games is a superset of the set of dyadic rationals, containing new objects like nimbers, infinitesimals, and switches, there are new properties and new operations that can be enacted on them. This chapter addresses some of those.

As usual, unless noted otherwise, assume that every game we see in this chapter is short.

6.1. Birthdays

You say it's your birthday
It's my birthday too, yeah
They say it's your birthday
We're gonna have a good time

- The Beatles, "Birthday"

Throughout Chapter 3, we saw that game values are defined recursively. That is, the game 1 is defined as $\left\{ 0 \,\middle|\, \right\}$, the game 2 as $\left\{ 1 \,\middle|\, \right\} = \left\{ \left\{ 0 \,\middle|\, \right\} \,\middle|\, \right\}$, $\uparrow =$

$\{ 0 \mid * \}$, etc. We can see that the definition of every game is, directly or indirectly, dependent on the game 0. So, if we imagine beginning with only the game 0, and then using it to generate all other games, then the act of creating games is also very much like a game itself.

We define the *birthday* of a game in the following way. The game 0 has birthday 0. Any game we can define using only 0, i.e. $1 = \{ 0 \mid \ \}$, $-1 = \{ \ \mid 0 \}$, and $* = \{ 0 \mid 0 \}$, have birthday 1. Any game that we can define using only $0, 1, -1$, and $*$, e.g. $\uparrow = \{ 0 \mid * \}$, $\{ 1 \mid -1 \}$, $2 = \{ 1 \mid \ \}$, etc. has birthday 2, and so on. Formally, we can write this using a recurrence relation like we introduced in Section 0.2.

$$b(G) = \begin{cases} 0 & \text{if } G = 0 \\ \max\{b(H)\}_{H \text{ an option of } G} + 1 & \text{otherwise} \end{cases}$$

Note that the birthday of a game is precisely the depth of its game tree. We sometimes say that a game is *born on* its birthday. So 0 is born on day 0; $1, -1$, and $*$ are born on day 1, and so on. Below is a table of the number of games with a given birthday. Try to list some more games with birthday 2.

We can find an upper bound for the number of games with birthday n without too much work. On day 1 we only have 0 and $\emptyset$ to work with. So if we choose one of these for L's option and one for R's, then we get an upper bound of $2 \cdot 2 = 4$ possible games born by the end of day 1. Since one of those games, 0, was born earlier, there are at most three games born on day 1. Now that we have four games the math gets a bit trickier. Remember that L can have more than one option, as can R. So we need to consider games like $\{ 0, * \mid 1 \}$. Since each player's options can be any subset of the games born on or before day 1, we need to count all subsets of the set of games born so far.

Math Diversion: Counting sets

We have already worked quite a bit with sets, which we introduced in Section 0.1. Let $S = \{a, b, c\}$ and recall that there are eight subsets of S:

$$\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \text{ and } S \text{ itself.}$$

We can do this exhaustively, but it gets exhausting if we want to consider larger sets. Let's instead count the number of subsets using some simple arithmetic.

Consider a group of four friends at a party. They're dressed in their nicest outfits and want to remember the occasion, so they decide to take pictures. But they don't just want a single group picture. Instead, each person wants a picture by themselves, with each other person, with each possible group of three, and with all four of them. Finally, because the sunset is so nice, they want a photo it without anyone in the shot. The number of photos they want to take is precisely the number of subsets of a set of four elements. We can count the number of photos by considering each person and whether or not they are in the shot, one photo at a time. For each photo, the first person is either in the shot or not, the second person is either in the shot or not, etc. So, the number of photos that need to be taken is equal to $2 \cdot 2 \cdot 2 \cdot 2 = 16$.

In general, if a finite set has n-many elements, then there are 2^n distinct subsets. How many subsets are there of a set of eight elements?

A: *There are $2^8 = 256$ distinct subsets.*

The number of games born by the end of day 2 is at most equal to the number of subsets of the set of games born by the end of day 1 squared, i.e. $(2^4)^2 = 256$. We can then subtract the four games we've already seen and get an upper bound of 252 for the number of games born *on* day 2. But this is a pretty steep overestimate. Note that we've included $\left\{ -1 \,\middle|\, 1 \right\}$ and $\{ * \mid * \}$, which are both equal to 0. In fact, there are only 18 new games born on day 2! Here is a table with the number of games born on each given day.

day	0	1	2	3
games born today	1	3	18	1452

You can see how quickly the number increases. There are clearly an infinite number of games. But *how infinite*?

So far we have dealt primarily with finite sets. The number of elements in a finite set S is its *cardinality*, written $|S|$, and given two finite sets it's very easy to tell whether or not their cardinalities are equal. $|\{a, b, c\}| = |\{x, y, z\}|$ because both are equal to 3. But we've been touching on infinite sets, as well. Remember $\mathbb{N} = \{0, 1, 2, \ldots\}$? We often refer to the set of natural numbers as the *counting numbers*. What is $|\mathbb{N}|$?

When we want to consider cardinalities of infinite sets, it's best to simply compare them to other sets we already know. Let's let $E = \{0, 2, 4, 6, \ldots\}$ be the set of all even natural numbers. It may seem at first glance that $|E| < |\mathbb{N}|$. After all, E only contains half of the elements from $\mathbb{N}$. But E is also infinite. Let's look closer at these two sets and try to compare them one element at a time.

One way to turn $\mathbb{N}$ into E is to eliminate every odd number. In that sense we are striking half of the terms from $\mathbb{N}$. But another method is to take every element in $\mathbb{N}$ and double it. In this case we haven't eliminated anything. In fact, the function $f : \mathbb{N} \to E$ defined by $f(n) = 2n$ is a *bijective function* or simply a *bijection*.

Math Diversion: Injective, surjective, and bijective functions

A *function* is a relationship from one set A to another set B, written $f : A \to B$, where every element a in the *domain* A has exactly one associated element $f(a)$ in the *codomain* B. The smallest subset of B that contains all elements of the form $f(a)$ for some $a \in A$ is called the *range* of f, sometimes written $f(A) \subseteq B$.

If $f(A) = B$ then we say that f is *onto* B, and we call f a *surjective* function. For example, let $A = \{a, b, c, d\}$, $B = \{x, y, z\}$ and consider the function $f : A \to B$ by the following table.

A	a	b	c	d
$f(A)$	y	x	z	x

This function f is surjective since every element of B has a *pre-image* in A.

Another example is the function $g : \mathbb{R} \to \mathbb{R}$ by $g(x) = 3x - 1$. To show this function is surjective, let y be any element of the codomain $\mathbb{R}$ and find an x in the domain $\mathbb{R}$ such that $g(x) = y$. In this case we can work backwards from $y = 3x - 1$ to find $x = \frac{1}{3}(y + 1)$. Since $g(x) = y$, and $\frac{1}{3}(y + 1) \in \mathbb{R}$ for any $y \in \mathbb{R}$, the function g is surjective.

As a final example, consider $h : \mathbb{R} \to \mathbb{R}$ by $h(x) = x^3 - x$. Working backwards in this case is very difficult. Instead, notice that h is a cubic polynomial, so we know that its range is all real numbers.

Is the function $f(x) = \ln(x) + 3$ surjective from $(0, \infty)$ to $\mathbb{R}$?

A.: *If* $y = \ln(x) + 3$, *then* $(y - 3) = \ln(x)$ *and thus* $x = e^{(y-3)}$. *Since* $e^{(y-3)} \in (0, \infty)$ *for all* $y \in \mathbb{R}$, f *is surjective.*

An *injective* or *one-to-one* function never sends the two different elements of a domain to the same element in the codomain. If $A = \{a, b, c, d\}$, $B = \{w, x, y, z\}$, and $j : A \to B$ by

A	a	b	c	d
$j(A)$	y	x	z	w

then j is injective because no two values in A go to the same element of B. Actually, this function is not only injective but also surjective. Any function that is both injective and surjective is called *bijective*.

As another example, consider the function $k : \mathbb{N} \to \mathbb{N}$ by $k(n) = 3n + 1$. To show this function is injective, consider two elements in A that have the same image, and demonstrate that they are equal.

If $n_1, n_2 \in A$ and $k(n_1) = k(n_2)$ then $3n_1 + 1 = 3n_2 + 1$. We can simplify this equation to $3n_1 = 3n_2$ and finally to $n_1 = n_2$. Since $k(n_1) = k(n_2)$ only when $n_1 = n_2$ we know that k is injective.

As an example of a function that is not injective, consider $m : \mathbb{R} \to \mathbb{R}$ by $m(x) = x^2 + 2x - 3$. If $x_1, x_2 \in \mathbb{R}$ and $m(x_1) = m(x_2)$ then $x_1^2 + 2x_1 - 3 = x_2^2 + 2x_2 - 3$. We could try to simplify this equation but we won't get far. Notice that $(2)^2 + 2(2) - 3 = 5 = (-4)^2 + 2(-4) - 3$. Since 2 and -4 are both in the domain $\mathbb{R}$ and $m(2) = m(-4)$ the function m is not injective.

Can you find a bijection between $\mathbb{Z}$ and $\mathbb{N}$?

A: *We need something both one-to-one and onto. Consider the following function.*

$$g(n) = \begin{cases} 0 & \text{if } n = 0 \\ 2n & \text{if } n > 0 \\ -2n - 1 & \text{if } n < 0 \end{cases}$$

We can see that no two integers project to the same natural number so g is injective. For any $y \in \mathbb{N}$, if y is even and positive then its pre-image is positive, if it's odd then its pre-image is negative, and if $y = 0$ then its pre-image is 0. So g is surjective. Therefore g is bijective.

If there is a bijection between two sets then we say they have the same cardinality. So $|\mathbb{N}| = |E|$! But that may not be surprising. After all, they're both infinite sets. In fact, Georg Cantor [3] showed that some infinite sets are actually larger than others using what has become known as his *diagonal argument*.

We say that any set with the same cardinality as $\mathbb{N}$, or less, is *countable*. So E, $\{a, b, c\}$, $\emptyset$, and $\mathbb{N}$ are all countable sets. But is every set countable? Consider the set $\mathbb{R}_{[0,1]}$ of all real numbers between 0 and 1. We will demonstrate that this set is *not* countable using a proof by contradiction.

Claim 6.1.1. *The set $\mathbb{R}_{[0,1]}$ is incountable.*

Proof. First, let's assume that $\mathbb{R}_{[0,1]}$ *is* countable. So we are assuming that there is a bijection between $\mathbb{N}$ and $\mathbb{R}_{[0,1]}$. That means that we assume there is a way to list the elements of $\mathbb{R}_{[0,1]}$ side-by-side with the elements of $\mathbb{N}$ in such a way that every element of each set appears exactly once. We can visualize this as a table like so:

$\mathbb{N}$	$\mathbb{R}_{[0,1]}$
0	$0.x_{0,0}x_{0,1}x_{0,2}x_{0,3}x_{0,4}x_{0,5}\dots$
1	$0.x_{1,0}x_{1,1}x_{1,2}x_{1,3}x_{1,4}x_{1,5}\dots$
2	$0.x_{2,0}x_{2,1}x_{2,2}x_{2,3}x_{2,4}x_{2,5}\dots$
3	$0.x_{3,0}x_{3,1}x_{3,2}x_{3,3}x_{3,4}x_{3,5}\dots$
4	$0.x_{4,0}x_{0,1}x_{0,2}x_{0,3}x_{0,4}x_{0,5}\dots$
$\vdots$	$\vdots$

where $x_{a,b}$ is the b^{th} digit of the number associated with the natural number a. We don't know what any of these digits are, but by our assumption they exist and every real number is on this list somewhere. Let's now consider the following real number $x = 0.x_0x_1x_2x_3\dots$ where we define the digits by

$$x_k = \begin{cases} 5 & \text{if } x_{k,k} = 7 \\ 7 & \text{otherwise} \end{cases}$$

In other words, we consider the boxed digits below and make certain that x does *not* match them.

$\mathbb{N}$	$\mathbb{R}_{[0,1]}$
0	$0.\boxed{x_{0,0}}x_{0,1}x_{0,2}x_{0,3}x_{0,4}x_{0,5}\cdots$
1	$0.x_{1,0}\boxed{x_{1,1}}x_{1,2}x_{1,3}x_{1,4}x_{1,5}\cdots$
2	$0.x_{2,0}x_{2,1}\boxed{x_{2,2}}x_{2,3}x_{2,4}x_{2,5}\cdots$
3	$0.x_{3,0}x_{3,1}x_{3,2}\boxed{x_{3,3}}x_{3,4}x_{3,5}\cdots$
4	$0.x_{4,0}x_{4,1}x_{4,2}x_{4,3}\boxed{x_{4,4}}x_{4,5}\cdots$
$\vdots$	$\vdots$

This number x, which is a real number, differs from every number on the list in at least one digit. So it's not on the list, and our assumption, that every real number *is* on the list, is wrong. Hence, $|\mathbb{R}| \neq |\mathbb{N}|$ and $\mathbb{R}$ is not countable. Therefore, there are infinite sets that are larger than others. □

Since we are now convinced that there are not only different finite cardinalities but also different *infinite* cardinalities, we can start to name them. We say that the cardinality of the naturals $|\mathbb{N}| = \aleph_0$. In general we let $\aleph_k$ represent the k^{th} infinite cardinality. So you may be expecting that $|\mathbb{R}| = \aleph_1$, but this is not necessarily the case! It's actually an open question whether or not there is some set S such that $|\mathbb{N}| < |S| < |\mathbb{R}|$, so instead we say that the *cardinality of the continuum* $|\mathbb{R}| = c$. It is also not true that every cardinality can be labeled with $\aleph_k$ for some $k \in \mathbb{N}$, since there are *far* more cardinalities than there are natural numbers. However, the question "What is the cardinality of the set of cardinalities?" is an interesting one, and after looking into Cantor's Theorem and the concept of a *proper class*, the interested reader will learn that the question as stated is not even valid.

The natural question is now which sets are countable? We know that E is and that $\mathbb{R}$ is not. What about the rationals $\mathbb{Q}$? Surely they must be bigger than $\mathbb{N}$.

As we've seen, if we can find a bijection between a set S and $\mathbb{N}$, or some ordering of S that includes every element exactly once, then S is countable. First, consider the following orderings of $\mathbb{Z}$ and $\mathbb{Z}^+$:

$\mathbb{N}$	0	1	2	3	4	5	…
$\mathbb{Z}$	0	1	-1	2	-2	3	…
$\mathbb{Z}^+$	1	2	3	4	5	6	…

You should be able to convince yourself that all elements of $\mathbb{Z}^+$ and $\mathbb{Z}$ appear exactly once in the list. We will use these lists to create an ordering, or *counting*, of $\mathbb{Q}$. Let the first rational be $\frac{0}{1}$, i.e. the first term in one list divided by the first term in the other. Then let the next two terms be $\frac{0}{2}$ and $\frac{1}{1}$, the first term in $\mathbb{Z}$ divided by the second term in $\mathbb{Z}^+$, and the second term in $\mathbb{Z}$ divided by the first term in $\mathbb{Z}^+$. Don't worry about reducing fractions just yet. Now let the next terms be $\frac{0}{3}, \frac{1}{2}, \frac{-1}{1}$. At each step we add a new numerator and a new denominator, slowly working our way through every possible pair from $\mathbb{Z} \times \mathbb{Z}^+$. We get the following list.

$\mathbb{N}$	0	1	2	3	4	…
$\mathbb{Q}$	$^0/_1$	~~$^0/_2$~~	$^1/_1$	~~$^0/_3$~~	$^1/_2$	…

We have crossed out terms that already appear on the list. We can remove these and shift everything up by one.

$\mathbb{N}$	0	1	2	3	4	…
$\mathbb{Q}$	$^0/_1$	$^1/_1$	$^1/_2$	$^{-1}/_1$	$^1/_3$	…
simplified	0	1	$^1/_2$	-1	$^1/_3$	…

Here we have a counting of the elements of $\mathbb{Q}$. You can convince yourself that every rational number appears once and only once in this list, and hence $|\mathbb{Q}| = \aleph_0$ and the rationals are countable. In fact, using this method, we can show that *any* countable collection of countable sets still only contains a countable set of elements. What does this mean for games? Well, we can order all short combinatorial games by their birthdays and use the resulting list to demonstrate that the set of all short games is countable.

$\mathbb{N}$	0	1	2	3	4	…
Games	0	1	-1	$*$	$\uparrow$	…

This brings us to our first *transfinite game*. Let's suppose that R has no options and L has as options every natural number. We end up with the game

$$\omega = \left\{\, 0, 1, 2, \ldots \,\middle|\, \right\}.$$

We can see that $\omega > 0$ and, in fact, $\omega > G$ for any short game G. This actually opens up the floodgates to the games

$$\begin{aligned} \omega + 1 &= \{\, \omega \mid \,\} \\ \omega + 2 &= \left\{\, \omega + 1 \,\middle|\, \right\} \\ &\vdots \end{aligned}$$

and even the infinitesimal

$$1/\omega = \left\{\, 0 \,\middle|\, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \cdots \,\right\}.$$

Note that if $r \in \mathbb{Q}^+$, then $1/\omega < r$, and it can be demonstrated that every infinitesimal $s > 0$ with a finite birthday is less than $1/\omega$.

Theorem 6.1.2. *$1/\omega$ is an infinitesimal and is at least as large as every infinitesimal with a finite birthday.*

Proof. We will prove both of these claims directly.

First, let $r \in \mathbb{Q}^+$ be any positive rational. There is some dyadic rational $1/2^n < r$. Consider the difference between $1/\omega$ and r.

$$\begin{aligned} g &= 1/2^n - 1/\omega \\ &= 1/2^n + \left\{\, -\frac{1}{2}, -\frac{1}{4}, -\frac{1}{8}, \cdots \,\middle|\, 0 \,\right\} \end{aligned}$$

Note that L moving first can go to $1/2^n - 1/2^{n+1} > 0$. Since L can win by going first, we know that $g \notin \mathcal{R}$. If R moves first, they can move to $1/2^n \in \mathcal{L}$ or to $1/2^{n-1} + 1/\omega$, to which L can respond by moving to $1/2^{n-1} - 1/2^n > 0$. Hence, $1/\omega \le 1/2^n < r$ for any positive rational r, and $1/\omega$ is an infinitesimal.

Next, consider any positive infinitesimal game ϵ born on day n, and let h be the following game.

$$h = 1/\omega - \epsilon$$

Moving first, L can move to $-\epsilon < 0$, or to $1/\omega - \epsilon^L$, where ϵ^L is a left option of ϵ born before day n. To the latter, R can move to $1/2^k - \epsilon^L > 0$ for some natural number k, or to $1/\omega - (\epsilon^L)^R$. Note that L can continue to play on ϵ as long as R does, and since, by Theorem 3.5.2, this game will eventually end with 0 and $\epsilon > 0$, eventually R must play on $1/\omega$ before L resulting in a positive number. Therefore, $h > 0$ and $1/\omega > \epsilon$ for any infinitesimal ϵ with a finite birthday. □

Now that we have the concept of birthdays we can present an interesting result that may help us to determine whether or not $G < H$ for a pair of games. First, we note that we can always find a positive game as small as we want.

Theorem 6.1.3. *If $G > 0$ then $0 < +_n < G$, where n is the birthday of G.*

Proof. Let H be the game $G - +_n = G + -_n = G + \left\{ \left\{ n \mid 0 \right\} \middle| 0 \right\}$. L can move to $G + \{ n \mid 0 \} > 0$. If R moves first, they can only move to $G^R + -_n$ or to G. In the former case, L can respond to $G^R + \{ n \mid 0 \}$. Since G^R is a right option of G, it is in $\mathcal{L}$ or $\mathcal{N}$ and its birthday is less than n, hence $n > G^R$. Whether R moves from here to $(G^R)^R + \{ n \mid 0 \}$ or to G^R, L can win. Therefore, $0 < +_n < G$. □

Theorem 6.1.3 leads us to the following Corollary.

Corollary 6.1.4. *If $H < G$ and $X = -H - +_n$, where n is the birthday of the game $G - H$, then $G + X > 0$ and $H + X < 0$*

Proof. The game $G + X = G - H - +_n$, which by Theorem 6.1.3 is positive. The game $H + X = H - H - +_n = -_n < 0$. □

7. CGT Beyond this Book

There are many combinatorial games topics we did not include in this book[1]. We'll mention these here and explain them briefly. Much of what we'll mention here is covered in other CGT texts:

- *Winning Ways for your Mathematical Plays*, a four-volume text by Elwyn Berlekamp, John H. Conway, and Richard Guy. This is the classic combinatorial games text, which includes many examples and illustrations.
- *Lessons in Play*, a text by Michael Albert, Richard Nowakowski, and David Wolfe. It is used in many advanced college-level CGT courses.
- *Combinatorial Game Theory*, a text by Aaron Siegel. This book covers many advanced topics in CGT.

Advanced topics that we don't cover here include (but are not limited to):

- Loopy games. There are rulesets where a position may be repeated in the course of play, meaning a game state may be a follower of itself.
- Reversible options. This allows us to improve analysis by replacing an option, H of G by H's grandchild-option when certain conditions are met.
- Octal games, a family of impartial rulesets involving heaps of stones. E.g. NIM and KAYLES.
- Game Transformations. Transforming game positions from one ruleset to another can be a way to see winning strategies that weren't evident before. It can also be used to prove that winnability is a computationally-intractible problem.

[1]Some we may include in future versions.

- Temperature strategies. The study of temperature and how to use it to pick a move goes much deeper than we covered here.

- Misère play. In this text we only covered normal play, which is when the player who makes the last play wins. Under Misère play, the last player to move loses instead.

- Scoring Games. In another alternative to normal play, players earn points in Scoring Games, then, when no more moves can be made, the player with the higher score wins. (Usually instead of having two separate scores, we just have a running total of the difference. L is winning if the score is positive; R if the score is negative.)

- Atomic Weights. The atomic weight of a game amounts to the number of moves one player could pass on a game, if played in isolation, and still make the last move. The game $\uparrow = \{\, 0 \mid * \,\} = \left\{ 0 \,\middle|\, \left\{ 0 \,\middle|\, \{\, 0 \mid 0 \,\} \right\} \right\}$ has atomic weight 1 since L can immediately win, or wait until after R's first move and then win. Similarly, $\Uparrow * = \left\{ 0 \,\middle|\, \{\, 0 \mid \uparrow \,\} \right\}$ has atomic weight 2.

- Alternative sums: The addition we've used throughout this text is the *disjunctive sum*. There are many other ways to add games that can also arise naturally. (E.g. ordinal sums in HACKENBUSH.)

Although what are considered "publishable" results in CGT are certainly subjective, current research topics can include the following. (Naturally, these need to be original results.)

- Creating a new interesting ruleset, and proving interesting properties about it. (These properties really do need to be interesting! A new ruleset on its own is often not enough.) These kinds of results are common, and that's okay!

- Increasing the known range of a ruleset, R, by finding values that were previously unknown to be in R. (For example, it is unknown whether there is a DOMINEERING position equal to $*4$.)

- Solving a well-known ruleset. (E.g. finding a trick to quickly determine the winnability or value.) This could either be a statement about all positions, or just the starting positions.
- Finding novel connections between some rulesets.
- Proving that positions in a ruleset are computationally intractible. (E.g. it is unknown whether CLOBBER is PSPACE-hard.)
- Anything novel about misère play. There's lots of research on this, especially in impartial games.
- And, of course, answering open problems posed in published combinatorial games papers.

There are certainly other avenues for results that we haven't included here.

A. Ruleset List

Here are the Rulesets used throughout this text, in alphabetical order.

AVOID TRUE

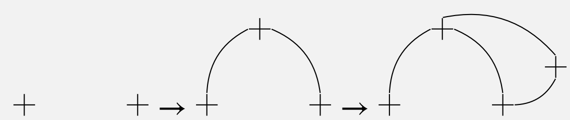
AVOID TRUE is a game played with a list of boolean variables $(x_1, x_2, \ldots, x_n)$ and a CNF, f, using those variables that has no negations. All variables begin the game set to False. A turn consists of picking one variable that is still False and flipping it to True, such that the whole formula still evaluates to False. (A variable cannot be chosen if flipping that would cause the formula to become True.)
To simplify positions, we will remove clauses that are already satisfied and list extra variables afterwards.

$$(x_1 \vee x_2) \wedge (x_2 \vee x_3 \vee x_4) = F$$
$$(x_1 \vee x_2) \wedge (x_2 \vee T \vee x_4) = F$$
$$(x_1 \vee x_2) \wedge (x_2 \vee T \vee T) = F$$

The first move is to flip x_3, making the whole second clause true. The second flips x_4, which leaves no further moves.

BRUSSELS SPROUTS

BRUSSELS SPROUTS is an impartial game played on a planar graph. Each vertex has four "arms" where edges can be attached. Each arm can only connect to one edge. A turn consists of creating a new vertex (with the four arms) then adding edges from two opposite arms of that new vertex to connect to previously-created arms. These new edges must maintain the planarity of the graph, which means that when drawn on paper, they may not cross any edges nor vertices. The chosen arms must not yet have attached edges. If there is no way to draw a new vertex with such arms, then there are no legal moves.
For simplicity of actual play, instead of drawing circles for the nodes, it is common to draw a "plus sign" to indicate the four arms, or just draw a short line segment across a line to signify the new node on that edge.

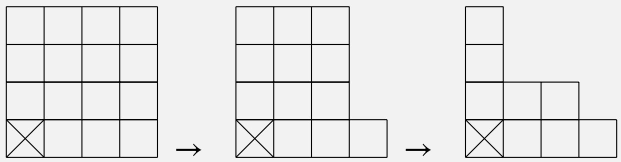


The first player adds the node at the top and draws the two edges. The second player, then connects that same node with the one on the right. The new node that's drawn can't connect on the left-hand-side.

Fact: The value of a BRUSSELS SPROUTS position is always either $*$ or 0.

CHOMP!

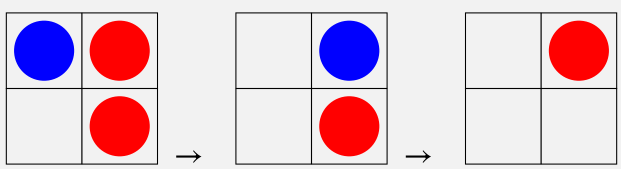
CHOMP (sometimes stylized *Chomp!*) is an impartial game played on a grid with squares labeled with integer coordinates from $[1, n] \times [1, m]$. On their turn a player chooses a remaining square with label (x, y) and removes it along with all remaining squares of the form (x_1, y_1) such that $x_1 \geq x$ and $y_1 \geq y$. The player who removes square $(1, 1)$ loses.



The first play is at $(4, 2)$, removing $(4, 2)$, $(4, 3)$, and $(4, 4)$. The next play is at $(2, 3)$, removing that position along with all pieces above and to the right of this piece.

CLOBBER

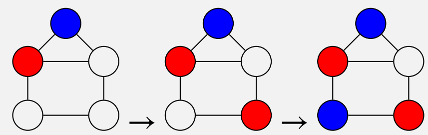
CLOBBER is a partisan game in which players take turns moving a piece of their color in any of the four orthogonal directions onto a piece of their opponent's color, removing the other's piece from play.



L moves right to take the red piece, then R moves up to win.

Col

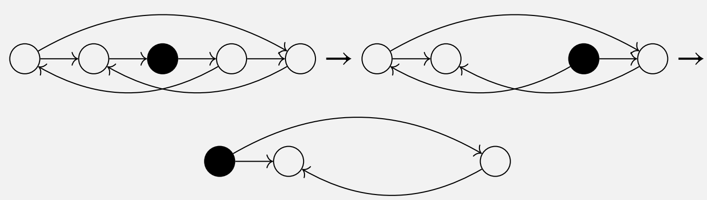
Col is a game played on a graph. L uses the color Blue, while R uses Red. Each turn, a player chooses an uncolored vertex, v, that is not adjacent to any vertices in their color, then paints v with their color.



R makes the first move, choosing the only vertex they can to color red. After that, L also has only one move, so they color that vertex blue.

Directed Geography

Directed Geography is an impartial game on an undirected graph, and a token on one of the vertices. A turn consists of moving the token along an outgoing edge to a new vertex, then removing the prior vertex from the graph. Directed Geography was first known as Genereralized Geography, as it is a generalization of Geography to graphs. It is also known as Directed Vertex Geography and often colloquially just as Geography.

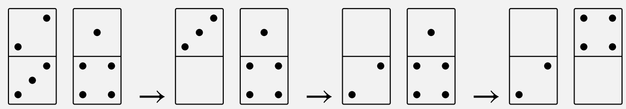


The token begins on the middle vertex. The first player moves it to the right, deleting the middle vertex they came from. The second player then uses the edge heading back to the left side.

Fact: This was one of the earliest games shown to be PSPACE-complete.

Dominim

Dominim is a Nim variant in which players are presented with a collection of dominoes. Each domino has a number of pips on top and a possibly different number on the bottom. Players make Nim moves on the set of domino tops, and once a domino is played it is flipped, so the bottom becomes the new top. The game ends when all tops contain zero pips.



The first player removes both pips from the lefthand domino then flips it. The second player responds by removing one from the same domino and flips it again, making in unplayable. The next player is forced to remove the only pip in the righthand domino, leaving the following player a single heap of four to remove and win.

Fact: If $top > bottom$ in a single domino game then its Grundy value is equal to $bottom + 1$.

Fibonacci Nim

Fibonacci Nim is played identically to Nim with two additional restrictions: The first player may not remove all the sticks, and no player may remove more than twice the number of sticks removed on the previous turn.

The first player removes 2 sticks. Since the second player can remove at most $2 \cdot 2 = 4$, they do so. The first player can remove the remaining $3 < 2 \cdot 4$.

GEOGRAPHY

GEOGRAPHY is an impartial ruleset in which players take turns naming towns, cities, or countries under the restrictions that no place may be named twice, and the next place named must begin with the last letter of the previously name place. "House" rules differ, sometimes allowing only cities or only countries to be named.

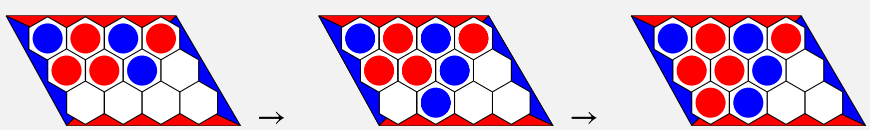
AntwerP → PlymoutH → HalifaX → XanadU

Beginning with Antwerp, subsequently named cities begin with the ending letter of the previous play.

Fact: If you're stuck on an X then consider Chinese cities.

HEX

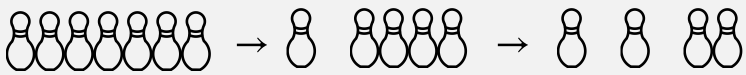
HEX is a partisan placement game played on a parallelogram-shaped board of hexagonal spaces. The four edges of the board are colored, blue on the left and right, and red on the top and bottom. Players take turns adding a single piece of their color, and the first to complete an unbroken line of monochromatic pieces of their color connecting their two sides of the board wins.



Left places a piece, then Right blocks for the win.

KAYLES

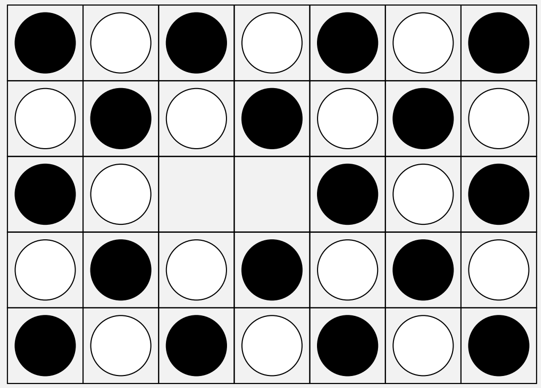
KAYLES is a bowling game created by Henry Dudeney in 1908[], derived from the lawn game Skittles. Each turn, players bowl a ball towards a row of bowling pins that may include some gaps. The ball can either knock over (remove) a single pin or two adjacent pins. Removed pins leave gaps in the row. The game ends when all pins have been removed.



The first bowl takes out the second and third pins, the second bowl removes only the fifth pin.

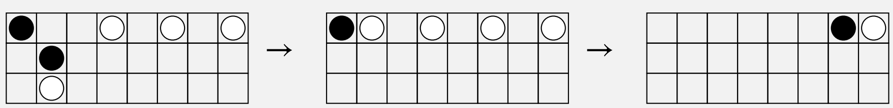
KONANE

KONANE is a traditional Hawaiian game that is similar in many ways to Checkers or Draughts. Black and white stones begin on a grid in an alternating fashion, with two adjacent stones removed:



Each turn, the current player uses one of their stones to jump over an opponent's orthogonally-adjacent piece to the empty spot on the other side. If there is another opponent piece (and empty space behind) along the same line, the jumping can continue if the player wishes.

GENERALIZED KONANE is a variant where starting stones don't need to alternate colors. In other words, two stones of the same color can be orthogonally adjacent.



R makes the first move by jumping their lower piece up. This sets up a big move for *L*, who makes a triple jump.

Nim

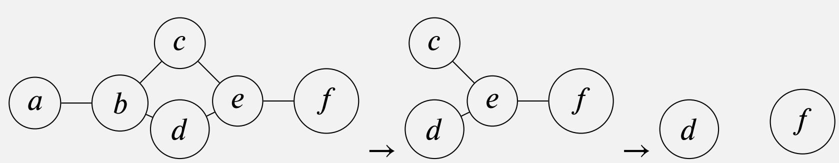
NIM is an ancient game that has been played in different forms across different cultures. In 1901, Charles Bouton coined the name “Nim” and described the complete theory[2].
The game play consists of removing objects (classically tokens, sticks, or pebbles) from piles (“heaps” or “Nim heaps”). Each turn, a player removes objects (at least one) from exactly one heap. The game ends when all heaps are empty.

\ /| /|| → / \/ → \ \

First move takes the entire last heap; the second takes one stick from the second.

Node Kayles

NODE KAYLES is a game played on an undirected graph. On the current player’s turn, they choose a vertex. Then, the graph is altered by removing that vertex and all adjacent vertices. When there are no more vertices, then there are no more moves.



The first move chooses a, the second chooses c.

SUBTRACTION

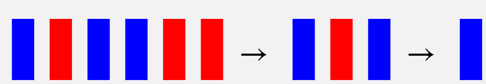
SUBTRACTION is a game played on a heap of n tokens, with a specified set of positive integers ($\mathbb{Z}^+$) known as the subtraction set. Each turn, the current player chooses a number k from the set such that $k \leq n$, and then k tokens are removed from the heap. When n is lower than all elements of the set, there are no more moves and the current player loses. We will describe each position using the fancy number script as before (e.g. 6) or by also including the subtraction set below if it's not understood from context (e.g. $\underset{\{1,5,6\}}{6}$).

$$\underset{\{1,2,4\}}{9} \rightarrow 7 \rightarrow 3 \rightarrow 1 \rightarrow 0$$

The first player takes two tokens to move to a heap of 7. The second player then takes four tokens to move to a heap of size 3. Taking three is not in the subtraction set, so the first player instead takes two, and the second player responds by taking the final token to win.

TOPPLINGDOMINOES

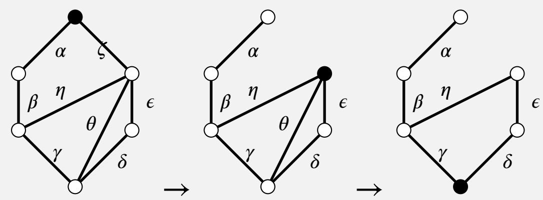
TOPPINGDOMINOES is a game played on rows of dominoes colored Red, Blue, or Green. Each turn, the current player picks either a green domino or domino of their color and a direction either left or right. (Either player can choose any direction.) The chosen domino and all other dominoes in the chosen direction are then "knocked down" and removed from play.



L makes the first move, knocking the fourth domino to the right. *R* makes the second move, knocking the second domino to the left.

Undirected Edge Geography

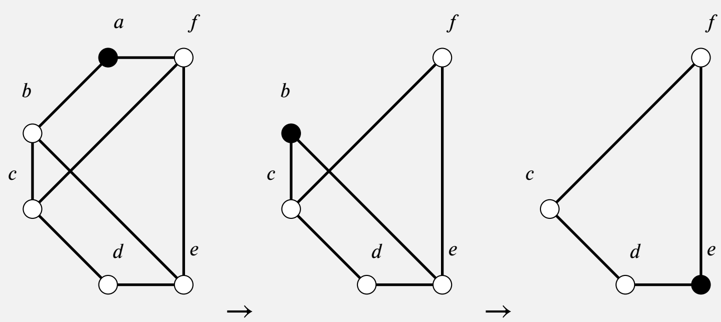
Undirected Edge Geography is played similarly to Undirected Vertex Geography, but with the modification that once an edge is played, only that edge is removed from play. Vertices remain.



The first player moves along edge ζ. The second player responds by moving along edge θ.

Undirected Vertex Geography

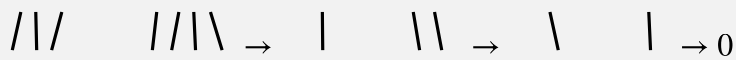
Undirected Vertex Geography is an impartial game on an undirected graph, wherein players take turns choosing a neighbor of the previously chosen vertex, then deleting that previous vertex from the graph.



Starting at vertex a, the first player moves to b and deletes a. The next player then moves to e and deletes vertex b.

WYTHOFF NIM

WYTHOFF NIM (also *Wythoff's Game*) is a two-heap NIM variant wherein players may remove $k > 0$ sticks from either pile or both piles at the same time.



The first player removes two sticks from both heaps. The second player then removes one to leave one in each. The first player responds by removing the sticks from both heaps at once.

B. Glossary

Glossary

follower A follower of G is a position that can be reached after one or more moves from G.. 6

impartial game tree A game tree for impartial games where edges from a position to its options are drawn as a fork extending downwards. All edges are drawn in the four cardinal directions (they may turn at perpendicular corners).. 4

option A position a player may move to.. 3

option notation The notation used to describe a partisan game by its options.. 96

partisan game A combinatorial game ruleset where players may have different options at some positions. (Note that this includes all impartial games as well. To differentiate, it's necessary to use the term *strictly* partisan, which are those rulesets without impartial games.. 95

C. Answers to Exercises

Here are the answers to starred exercises in the book.

Answer of exercise 0.1.0

Question:

$S = \{x \in \mathbb{N} \mid x > 5 \text{ and } x < 10\}$. Rewrite S without using set-builder notation.

⋆ Answer:

$S = \{6, 7, 8, 9\}$

Answer of exercise 0.1.1

Question:

$S = \{2k + 1 \mid k \in \mathbb{N} \cup \{0\} \text{ and } k \leq 6\}$. Rewrite S without using set-builder notation.

⋆ Answer:

$S = \{1, 3, 5, 7, 9, 11, 13\}$

Answer of exercise 0.1.2

Answer Not Provided

Answer of exercise 0.1.3

Answer Not Provided

Answer of exercise 0.1.4

Answer Not Provided

Answer of exercise 0.1.5

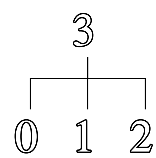
Answer Not Provided

Answer of exercise 0.1.6

Question:

Draw the first two levels of the impartial game tree from $\underset{\{1,2,3\}}{\mathbb{3}}$. (Your diagram should show the initial position and all the options from that position.)

⋆ Answer:



Answer of exercise 0.1.7

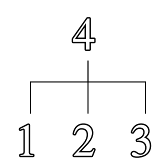
Answer Not Provided

Answer of exercise 0.1.8

Question:

Draw the first two levels of the impartial game tree from $\underset{\{1,2,3\}}{\mathbb{4}}$.(Your diagram should show the initial position and all the options from that position.)

⋆ Answer:



Answer of exercise 0.1.9

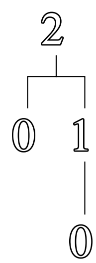
Answer Not Provided

Answer of exercise 0.1.10

Question:

Draw the entire impartial game tree for $\underset{\{1,2,3\}}{\mathbb{2}}$. Is there a winning move for the first player? Justify your answer. (This is a continuation of 0.1.7 .)

⋆ Answer:



Yes, there is a winning move for the first player, because you can move to a pile of 0.

Answer of exercise 0.1.11

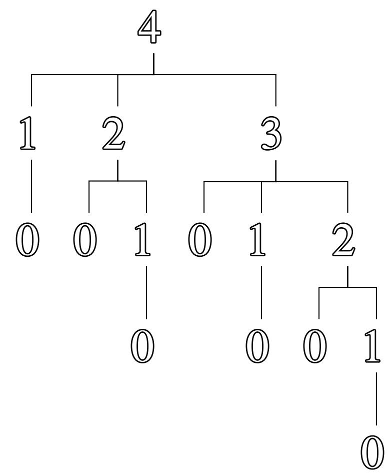
Answer Not Provided

Answer of exercise 0.1.12

Question:

Draw the impartial game tree for $\underset{\{1,2,3\}}{\mathbb{4}}$. Which of the two players (first or second) has a winning strategy from the pile of 4? Justify your answer. (This is a continuation of exercise 0.1.8.)

⋆ Answer:



The second player has a winning strategy, because no matter what the first player does, the second player can move to 0.

Answer of exercise 0.1.13

Answer Not Provided

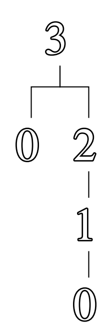
Answer of exercise 0.1.14

Question:

What are all of the followers of $\underset{\{1,3\}}{3}$?

⋆ Answer:

We can solve this by drawing out the impartial game tree from $\underset{\{1,3\}}{3}$:



The followers of $\mathbb{3}$ are $\mathbb{2}$, $\mathbb{1}$, and $\mathbb{0}$.

Answer of exercise 0.1.15

Answer Not Provided

Answer of exercise 0.1.16

Answer Not Provided

Answer of exercise 0.1.17

Question:

If $x \equiv 3 \pmod 5$ and $y \equiv 1 \pmod 5$, then what is $x + y \pmod 5$?

⋆ Answer:

$x + y \equiv 4 \pmod 5$

Answer of exercise 0.1.18

Answer Not Provided

Answer of exercise 0.1.19

Question:

How do a natural number and its square compare under arithmetic $\pmod 2$? i.e. are n and n^2 always equivalent $\pmod 2$, always non-equivalent $\pmod 2$, or sometimes equivalent and sometimes not?

⋆ Answer:

If n is even then $n \equiv 0 \pmod 2$, and since the square of an even is also even we get that $n^2 \equiv 0 \pmod 2$. Similarly, if $n \equiv 1 \pmod 2$ then $n^2 \equiv 1 \pmod 2$.

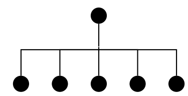
Answer of exercise 0.1.20

Answer Not Provided

Answer of exercise 0.2.0

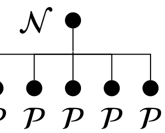
Question:

What is the outcome class at the root of this impartial game tree?



Justify your answer by labeling the nodes of the tree.

⋆ Answer:



The overall outcome class is: $\mathcal{N}$

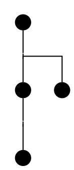
Answer of exercise 0.2.1

Answer Not Provided

Answer of exercise 0.2.2

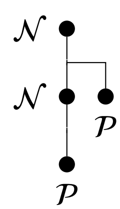
Question:

What is the outcome class at the root of this impartial game tree?



Justify your answer by labeling each node of the tree with its outcome class.

⋆ Answer:



The overall outcome class is: $\mathcal{N}$

Answer of exercise 0.2.3

Answer Not Provided

Answer of exercise 0.2.4

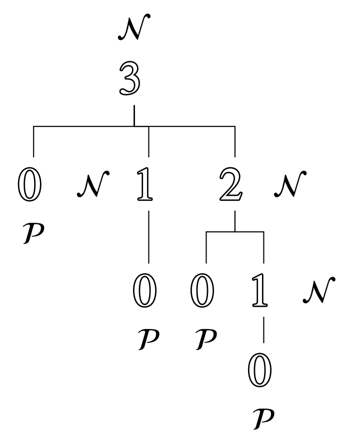
Answer Not Provided

Answer of exercise 0.2.5

Question:

Let $G = \underset{\{1,2,3\}}{\mathbb{3}}$. What is the $o(G)$? Justify your answer by drawing the game tree and labeling each node with its outcome class. (This is a continuation of exercise 0.1.11.)

⋆ Answer:



Thus, $o(G) = \mathcal{N}$.

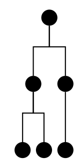
Answer of exercise 0.2.6

Answer Not Provided

Answer of exercise 0.2.7

Question:

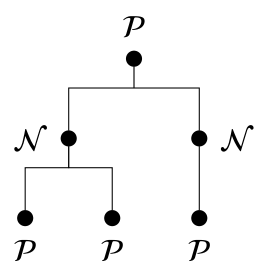
Find the outcome class of this tree:



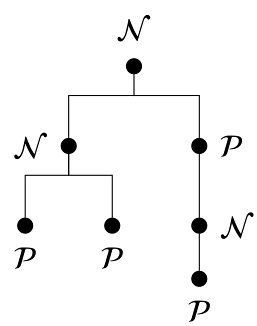
Then add one new child to one of the leaves to flip the outcome class at the root of the tree. (Yes, you need to show the work to derive the outcome class of the new tree.)

⋆ Answer:

The outcome class of the tree is currently $\mathcal{P}$:



However, if we add one new leaf to the bottom right-hand node, then the outcome class becomes $\mathcal{N}$:



Answer of exercise 0.2.8

Answer Not Provided

Answer of exercise 0.2.9

Question:

Verify that the `max` function in this chapter requires $6n+5$ Python instructions to run. Assume that the `len` function takes one step to complete.

⋆ Answer:

- Line 1 (`maximum = numbers[0]`) will be executed once. (1)
- Line 2: 1
- Line 3: The condition will be checked $n+1$ times and `len` will be called each time, so this contributes $2n+2$ towards the total.
- Line 4: This assignment happens n times.
- Line 5: n times. In the worst case, it is always true
- Line 6: n
- Line 7: n
- Line 8: After the loop, we will return exactly once, so this contributes only 1 step.

$1+1+2n+2+n+n+n+n+1 = 6n+5$.

Answer of exercise 0.2.10

Answer Not Provided

Answer of exercise 0.2.11

Answer Not Provided

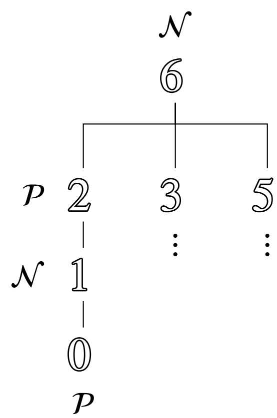
Answer of exercise 0.3.0

Question:

Use a trimmed game tree to find and show the outcome class of $\underset{\{1,3,4\}}{\mathbb{6}}$. (This is a continuation of exercise 0.1.9.) What is the smallest tree you can draw that proves your result?

⋆ Answer:

The position is in $\mathcal{N}$.



Answer of exercise 0.3.1

Answer Not Provided

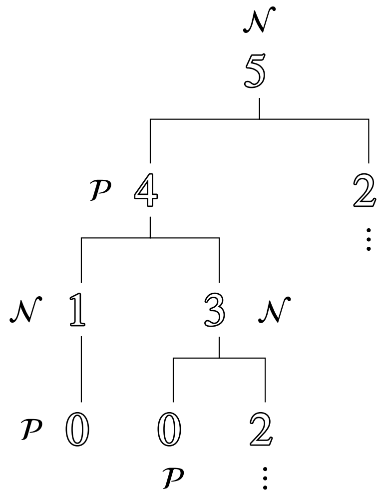
Answer of exercise 0.3.2

Question:

Use a trimmed game tree to find and show the outcome class of $\underset{\{1,3\}}{\mathbb{5}}$.

⋆ Answer:

The position is in $\mathcal{N}$.



Answer of exercise 0.3.3

Answer Not Provided

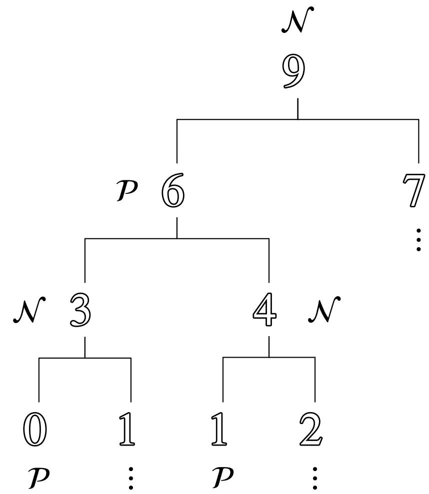
Answer of exercise 0.3.4

Question:

Use a trimmed game tree to find and show the outcome class of $\underset{\{2,3\}}{9}$.

⋆ Answer:

The position is in $\mathcal{N}$.



Answer of exercise 0.3.5

Answer Not Provided

Answer of exercise 0.3.6

Answer Not Provided

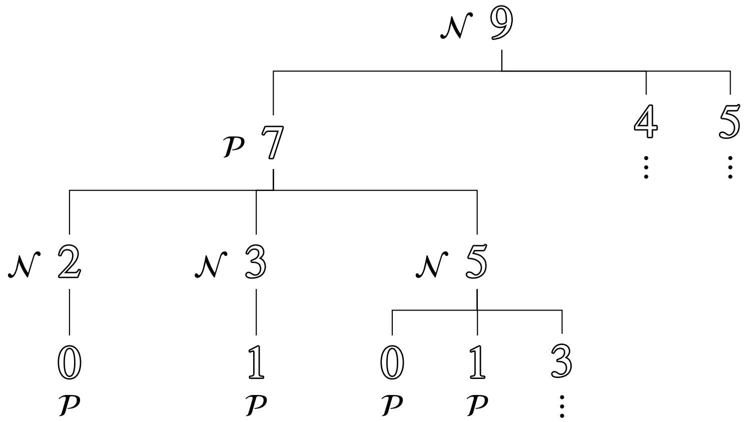
Answer of exercise 0.3.7

Question:

Use a trimmed game tree to find and show the outcome class of $\mathbb{9}_{\{2,4,5\}}$.

⋆ Answer:

The position is in $\mathcal{N}$.

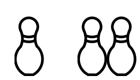


Answer of exercise 0.3.8

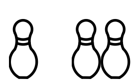
Answer Not Provided

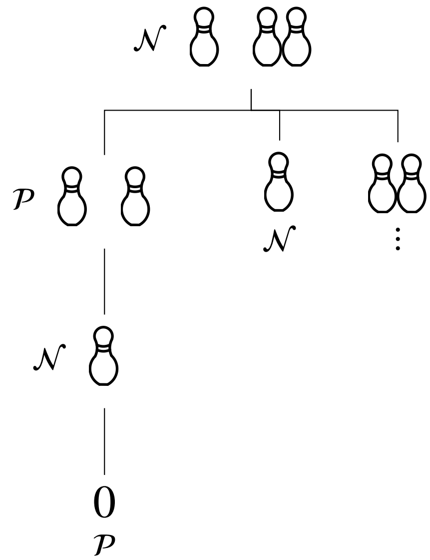
Answer of exercise 0.4.0

Question:

Use a (partial) game tree to find the outcome class of  .

⋆ Answer:

 is in $\mathcal{N}$.



Answer of exercise 0.4.1

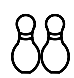
Answer Not Provided

Answer of exercise 0.4.2

Answer Not Provided

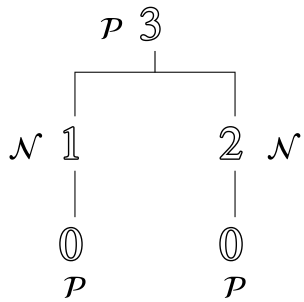
Answer of exercise 0.4.3

Question:

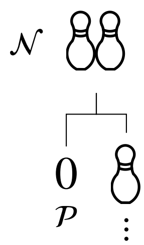
Find the outcome class of $\mathbb{3}_{\{1,2\}}$ +  . If possible, find and use the outcome classes for the two components and use that to justify your answer. If that fails, draw the game tree for the sum.

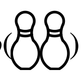
⋆ Answer:

First we find $o(\mathbb{3}_{\{1,2\}})$:



Next we find $o($$)$:



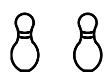
Since $o(\underset{\{1,2\}}{\mathbb{3}}) \in \mathcal{P}$ and $o($$) \in \mathcal{N}$, the sum is in $\mathcal{N}$.

Answer of exercise 0.4.4

Answer Not Provided

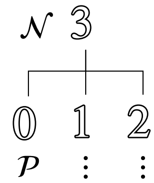
Answer of exercise 0.4.5

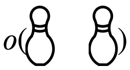
Question:

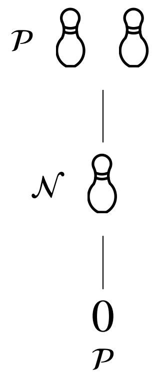
Find the outcome class of $\underset{\{1,2,3\}}{\mathbb{3}}$ +  . If possible, find and use the outcome classes for the two components and use that to justify your answer. If that fails, draw the game tree for the sum.

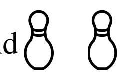
⋆ Answer:

First we find $o(\underset{\{1,2,3\}}{\mathbb{3}})$:



Next we find $o($$)$:



$\underset{\{1,2,3\}}{\mathbb{3}} \in \mathcal{N}$ and  $\in \mathcal{P}$, so the sum is in $\mathcal{N}$.

Answer of exercise 0.4.6

Answer Not Provided

Answer of exercise 0.4.7

Answer Not Provided

Answer of exercise 0.4.8

Answer Not Provided

Answer of exercise 0.4.9

Answer Not Provided

Answer of exercise 0.4.10

Question:

Prove that if games G and H are both in $\mathcal{P}$, then the game $G + H$ is in $\mathcal{P}$.

⋆ Answer:

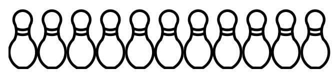
Any move on either game moves that game into an $\mathcal{N}$ position, which can be countered back to an $\mathcal{P}$ position. This can be continued until no moves remain in either game.

Answer of exercise 0.4.11

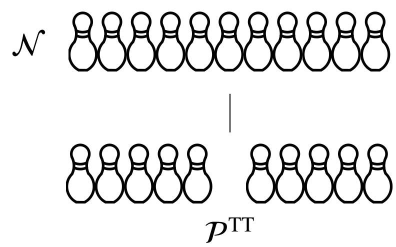
Answer Not Provided

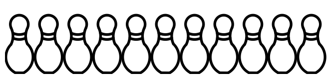
Answer of exercise 0.5.0

Question:

Find and prove the outcome class of  , a KAYLES row of 11 pins.

⋆ Answer:



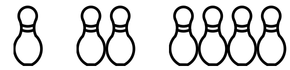
So,  is an $\mathcal{N}$-position.

Answer of exercise 0.5.1

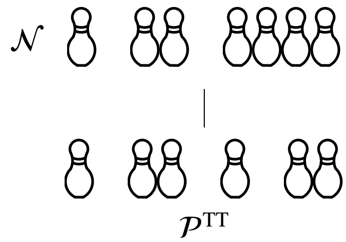
Answer Not Provided

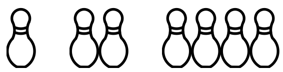
Answer of exercise 0.5.2

Question:

Find and prove the outcome class of  .

★ Answer:



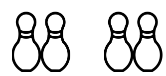
So,  is an $\mathcal{N}$-position.

Answer of exercise 0.5.3

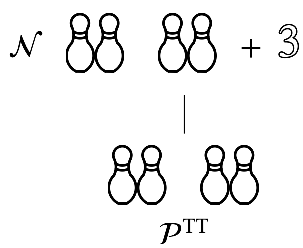
Answer Not Provided

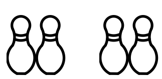
Answer of exercise 0.5.4

Question:

Find and prove the outcome class of  $+ \underset{\{1,3\}}{\mathbb{3}}$.

★ Answer:



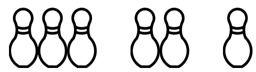
So,  $+ \underset{\{1,3\}}{\mathbb{3}}$ is an $\mathcal{N}$-position.

Answer of exercise 0.5.5

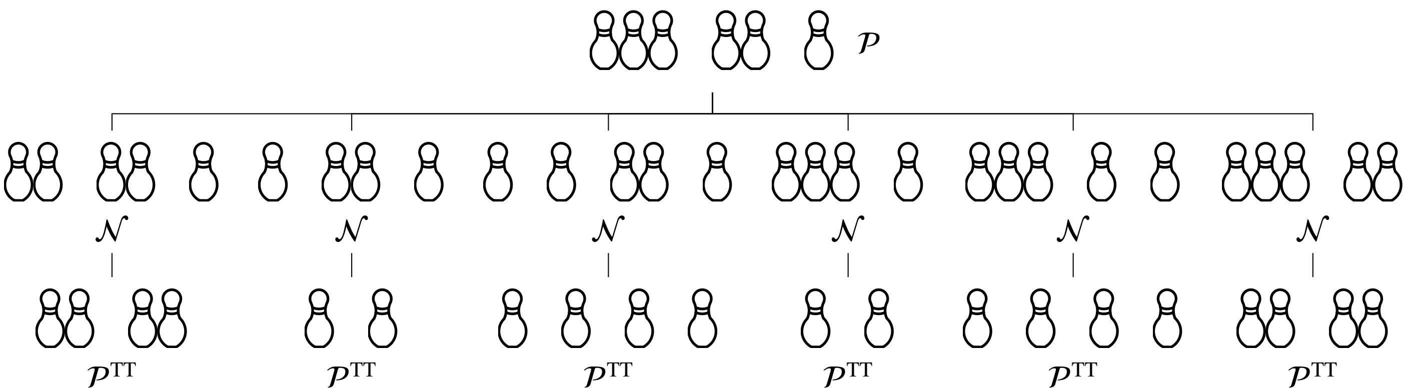
Answer Not Provided

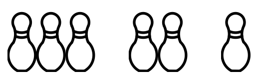
Answer of exercise 0.5.6

Question:

Find and prove the outcome class of  .

⋆ Answer:



So,  is a $\mathcal{P}$-position.

Answer of exercise 0.5.7

Answer Not Provided

Answer of exercise 0.6.0

Answer Not Provided

Answer of exercise 0.6.1

Answer Not Provided

Answer of exercise 0.6.2

Answer Not Provided

Answer of exercise 0.6.3

Answer Not Provided

Answer of exercise 0.6.4

Answer Not Provided

Answer of exercise 0.6.5

Answer Not Provided

Answer of exercise 0.6.6

Answer Not Provided

Answer of exercise 0.6.7

Answer Not Provided

Answer of exercise 0.6.8

Question:

Conjecture a formula for $\sum_{j=1}^{n} 2^j$ and prove your claim using induction.

★ Answer:

First, note that plugging in values for n yields $2, 6, 14, 30, 62, \ldots$, which all appear to be two less than powers of two. So we claim that $\sum_{j=1}^{n} 2^j = 2^{n+1} - 2$.

Proof. We proceed by induction on n. For the base case, consider that $\sum_{j=1}^{1} 2^j = 2 = 2^{1+1} - 2$.

Now assume that $\sum_{j=1}^{n} 2^j = 2^{n+1} - 2$. The sum

$$\begin{aligned}\sum_{j=1}^{n+1} 2^j &= 2^{n+1} + \sum_{j=1}^{1} 2^j \\ &= 2^{n+1} + 2^{n+1} - 2 \\ &= 2 \cdot 2^{n+1} - 2 \\ &= 2^{n+2} - 2 \qquad\qquad = 2^{n+1+1} - 2\end{aligned}$$

Therefore the claim is true. □

Answer of exercise 0.6.9

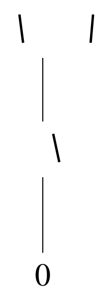
Answer Not Provided

Answer of exercise 0.6.10

Question:

Draw the full game tree for  .

★ Answer:



Answer of exercise 0.6.11

Answer Not Provided

Answer of exercise 0.6.12

Question:

What is $mex\left(\{0,1,2,3,4\}\right)$?

⋆ Answer:

$mex\left(\{0,1,2,3,4\}\right) = 5$

Answer of exercise 0.6.13

Answer Not Provided

Answer of exercise 0.6.14

Question:

What is $mex\left(\{0,1,2,4\}\right)$?

⋆ Answer:

$mex\left(\{0,1,2,4\}\right) = 3$

Answer of exercise 0.6.15

Answer Not Provided

Answer of exercise 0.6.16

Answer Not Provided

Answer of exercise 0.6.17

Answer Not Provided

Answer of exercise 0.6.18

Question:

If $S = \{0,1,2,3,4\}$, and $T = \{5,6,7,8,9\}$ what is $mex\left(S \cup T\right)$?

⋆ Answer:

$$\begin{aligned} mex\left(S \cup T\right) &= mex\left(\{0,1,2,3,4\} \cup \{5,6,7,8,9\}\right) \\ &= mex\left(\{0,1,2,3,4,5,6,7,8,9\}\right) \\ &= 10 \end{aligned}$$

Answer of exercise 0.6.19

Answer Not Provided

Answer of exercise 0.6.20

Answer Not Provided

Answer of exercise 0.6.21

Question:

If $S = \{0, 1, 2, 5, 6, 7\}$, and $T = \{0, 1, 4, 5, 8, 9\}$ what is $mex\left(S \cap T\right)$?

⋆ Answer:

$$\begin{aligned} mex\left(S \cap T\right) &= mex\left(\{0, 1, 2, 5, 6, 7\} \cap \{0, 1, 4, 5, 8, 9\}\right) \\ &= mex\left(\{0, 1, 5\}\right) \\ &= 2 \end{aligned}$$

Answer of exercise 0.6.22

Answer Not Provided

Answer of exercise 0.6.23

Answer Not Provided

Answer of exercise 0.6.24

Question:

If $S = \{0, 2, 4, 6, 8\}$, what is $mex\left(S^C\right)$?

⋆ Answer:

$$\begin{aligned} mex\left(S^C\right) &= mex\left(\{0, 2, 4, 6, 8\}^C\right) \\ &= mex\left(\{1, 3, 5, 7, 9\} \cup \{k \mid k \geq 10\}\right) \\ &= 0 \end{aligned}$$

Answer of exercise 0.6.25

Answer Not Provided

Answer of exercise 0.6.26

Answer Not Provided

Answer of exercise 0.6.27

Answer Not Provided

Answer of exercise 0.6.28

Question:

If $S = \{2k \mid k \in \mathbb{N}\}$, what is $mex\left(S\right)$?

⋆ Answer:

$S = \{0, 2, 4, \ldots\}$, so $mex\left(S\right) = 1$.

Answer of exercise 0.6.29

Answer Not Provided

Answer of exercise 0.6.30

Answer Not Provided

Answer of exercise 0.6.31

Question:

What is $mex\left(\mathbb{N} \setminus \{56\}\right)$?

⋆ Answer:

The set contains every element of $\mathbb{N}$ except for 56, so the mex is 56.

Answer of exercise 0.6.32

Answer Not Provided

Answer of exercise 0.6.33

Question:

Let $S = \{5k - 1 \mid k \in \mathbb{N}\}$. What is $mex\left(\mathbb{N} \setminus S\right)$?

⋆ Answer:

The set contains all natural numbers that are one less than a multiple of 5. That includes 4, 9, 14, etc. That means that the lowest natural not in S is 4, so the mex is 4.

Answer of exercise 0.6.34

Answer Not Provided

Answer of exercise 0.6.35

Question:

Let $S = \{2k \mid k \in \mathbb{N}\}$ and $T = \{5k \mid k \in \mathbb{N}\}$. What is $mex\left((S \setminus T)^C\right)$?

⋆ Answer:

The elements of $S \setminus T$ below 10 are: $\{2, 4, 6.8\}$, so the complement of that includes 0 and 1, but not 2. Thus, the mex is 2.

Answer of exercise 0.6.36

Answer Not Provided

Answer of exercise 0.6.37

Question:

Simplify $^*\left\{\ *, *3, *5, *7, *9\ \right\}$ to a single nimber value.

⋆ Answer:

The game is equal to 0 because $mex\left(1, 3, 5, 7, 9\right) = 0$.

Answer of exercise 0.6.38

Answer Not Provided

Answer of exercise 0.6.39

Question:

Simplify $^*\left\{\ 0, *, 0, 0, *\ \right\}$ to a single nimber value.

⋆ Answer:

The game is equal to $*2$ because $mex\left(0, 1, 0, 0, 1\right) = mex\left(0, 1\right) = 2$.

Answer of exercise 0.6.40

Answer Not Provided

Answer of exercise 0.6.41

Question:

$G = {}^{*}\left\{\ 0, *2, *4, {}^{*}\left\{\ 0, *2, *4\ \right\}\ \right\}$ includes another impartial game's options written out. Simplify this to a single nimber value.

⋆ Answer:

$$
\begin{aligned}
G &= {}^{*}\left\{\ 0, *\,2, *\,4, {}^{*}\left\{\ 0, *2, *4\ \right\}\ \right\} \\
&= {}^{*}\left\{\ 0, *2, *4, *\ \right\} \\
&= *3
\end{aligned}
$$

Answer of exercise 0.6.42

Answer Not Provided

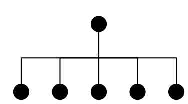
Answer of exercise 0.6.43

Answer Not Provided

Answer of exercise 0.6.44

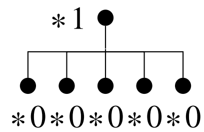
Question:

What is the value at the root of this impartial game tree?



Justify your answer by labeling each node of the tree with its value. (This is a follow-up to exercise 0.2.0.)

⋆ Answer:



The overall outcome class is: $\mathcal{N}$

Answer of exercise 0.6.45

Answer Not Provided

Answer of exercise 0.6.46

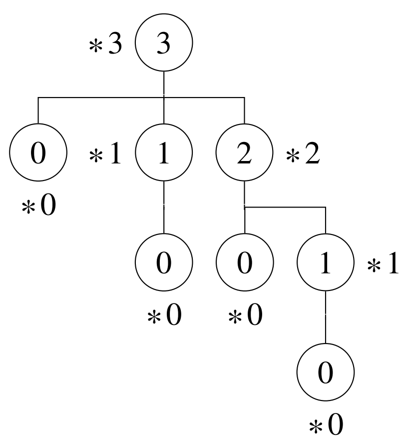
Answer Not Provided

Answer of exercise 0.6.47

Question:

Let G be the position in SUBTRACTION-$\{1, 2, 3\}$ with a pile of 3. What is the nimber of G? Justify your answer by drawing the game tree and labeling each node with its nimber. (This is a continuation of exercise 0.2.5.)

⋆ Answer:



Thus, $o(G) = \mathcal{N}$.

Answer of exercise 0.6.48

Answer Not Provided

Answer of exercise 0.6.49

Question:

Using the table for $\underset{\{1,2\}}{\mathbb{k}}$ as a model, create a similar table to find the nimber of $\underset{\{1,2,3\}}{\mathbb{5}}$.

★ Answer:

k	0	1	2	3	4	5
$\underset{\{1,2,3\}}{\mathbb{k}}$	0	$*$	$*2$	$*3$	0	$*$

Answer of exercise 0.6.50

Answer Not Provided

Answer of exercise 0.6.51

Answer Not Provided

Answer of exercise 0.6.52

Question:

Do the same as exercises 0.6.50 and 0.6.51 , except with the subtraction set $\{1,3\}$ instead.

★ Answer:

k	0	1	2	3	4	5	6	7	8	9	10
$\underset{\{1,3\}}{\mathbb{k}}$	0	$*$	0	$*$	0	$*$	0	$*$	0	$*$	0

The pattern seems to be: $\underset{\{1,3\}}{\mathbb{k}} = \begin{cases} 0, & k \equiv 0 \bmod 2 \\ *, & k \equiv 1 \bmod 2 \end{cases}$

$= * \, (k \bmod 2)$

Proof that the pattern works:

Claim: $\underset{\{1,3\}}{\mathbb{k}} = * \, (k \bmod 2)$

Proof. Proof by strong induction:

- **Base Cases**: $k = 0, 1$, and 2
  - $k = 0$: This is a terminal position, so the value is 0. ✓
  - $k = 1$: $\underset{\{1,3\}}{\mathbb{1}} = {}^*\left\{ \underset{\{1,3\}}{\mathbb{0}} \right\} = {}^*\left\{ 0 \right\} = *$. ✓
  - $k = 2$: $\underset{\{1,3\}}{\mathbb{2}} = {}^*\left\{ \underset{\{1,3\}}{\mathbb{1}} \right\} = {}^*\{ * \} = 0$. ✓
- **Recursive Case**: Let $i \geq 3$. Assume that $\forall k \in \{0, 1, \ldots, i-1\}$: $\underset{\{1,3\}}{\mathbb{k}}$ $=* (k \bmod 2)$

  Consider the case where $k = i$. Since $i \geq 3$, both moves are valid on $\underset{\{1,3\}}{\mathbb{i}}$.
  Thus,

$$
\begin{aligned}
\underset{\{1,3\}}{\mathbb{i}} &= {}^*\left\{ \underset{\{1,3\}}{\mathbb{i}\text{-}\mathbb{3}}, \underset{\{1,3\}}{\mathbb{i}\text{-}\mathbb{1}} \right\} \\
&= {}^*\left\{ * \left((i-3) \bmod 2\right), * \left((i-1) \bmod 2\right) \right\} \\
&= {}^*\left\{ * \left((i-1) \bmod 2\right), * \left((i-1) \bmod 2\right) \right\} \\
&= {}^*\left\{ * \left((i-1) \bmod 2\right) \right\} \\
&= \{0, *\} \setminus \{* \left((i-2) \bmod 2\right)\}
\end{aligned}
$$

  The mex of that set is then $(i-2) \bmod 2 = i \bmod 2$.

  Thus, ${}^*\left\{ * \left((i-1) \bmod 2\right) \right\} =* (i \bmod 2)$. ✓

□

Answer of exercise 0.6.53

Answer Not Provided

Answer of exercise 0.6.54

Question:

The version of the mex function given in the chapter is elegant partly because it is so short. Unfortunately, $O(n^2)$ is pretty inefficient. We can drastically speed it up to $O(n\log(n))$ by writing a bit more code. Rewrite mex to run in $O(n\log(n))$ time. (Hint: Python's built-in sort method for lists takes $O(n\log(n))$ time.)

⋆ Answer:

```
def mex(integers):
    '''Returns the mex (minimum excluded value) of integers.'''
    integers.sort() #lowest to highest
    num = 0
    for integer in integers:
        if integer < num:
            pass
        elif integer == num:
            num += 1
        else:
            return num
    return num
```

Answer of exercise 0.6.55

Answer Not Provided

Answer of exercise 0.6.56

Question:

In the text, we showed two different games, G_1 and G_2 that were both equal to $*4$. We showed that $G_1 + G_2 \in \mathcal{P}$, even though they weren't identical positions. Show that this works for any two equal (but not necessarily identical) positions. Prove that if $G = H$, then $G + H \in \mathcal{P}$.

⋆ Answer:

To show that $G + H \in \mathcal{P}$, we will show a winning strategy for the second player on $G + H$. If G and H have no options (so the sum is $0 + 0 = 0$) and the sum is already in $\mathcal{P}$. If either has options, then after the second player's turn, the game still has a new position $I + J$ where $I = J$.

Since $G = H$, by our definition of equality (for impartial games) it means that there is a nimber, k, such that $G =* k = H$. Thus, neither G nor H has an option equal to $*k$. Let's assume the first player chooses to move on G to a new position, I with value $*j$. There are two cases: $j > k$ and $j < k$.

Case 1, $j > k$: Then I has an option with value k. The second player can move to that option and the sum is still $*k + *k$ as needed.

Case 2, $j < k$: Then H has an option with value j. The second player can move to that option and the sum is still $*j + *j$ as needed.

If the first player plays on H instead of G, then we can use the same strategy, but with the roles of G and H switched. In these cases of proofs, it is common to say "Without loss of generality, assume …" before the assumption we're making. The meaning of this is that we're not making any special assumptions and the same proof works for both cases (but with the names swapped around).

Since the second player can always follow their strategy, the games will continue until they have no options and the second player wins. Thus, $G + H \in P$.

Answer of exercise 0.6.57

Answer Not Provided

Answer of exercise 0.7.0

Question:

What is the nimber value $*3 + *2$?

⋆ Answer:

$$
\begin{aligned}
*3 + *2 &= *(3 \oplus 2) \\
&= *(11_2 \oplus 10_2) \\
&= *(01_2) \\
&= *1 \\
&= *
\end{aligned}
$$

Answer of exercise 0.7.1

Answer Not Provided

Answer of exercise 0.7.2

Question:

What is the nimber value $*4 + *2$?

⋆ Answer:

$$\begin{aligned} *4 + *2 &= *(4 \oplus 2) \\ &= *(100_2 \oplus 10_2) \\ &= *(110_2) \\ &= *6 \end{aligned}$$

Answer of exercise 0.7.3

Answer Not Provided

Answer of exercise 0.7.4

Question:

What is the nimber value $*5 + *11$?

⋆ Answer:

$$\begin{aligned} *5 + *11 &= *(5 \oplus 11) \\ &= *(101_2 \oplus 1011_2) \\ &= *(0101_2 \oplus 1011_2) \\ &= *(1110_2) \\ &= *14 \end{aligned}$$

Answer of exercise 0.7.5

Answer Not Provided

Answer of exercise 0.7.6

Question:

What is the nimber value $*20 + *31$?

⋆ Answer:

$$\begin{aligned} *20 + *31 &= *(20 \oplus 31) \\ &= *(10100_2 \oplus 11111_2) \\ &= *(01011_2) \\ &= *(8 + 2 + 1) \\ &= *11 \end{aligned}$$

Answer of exercise 0.7.7

Answer Not Provided

Answer of exercise 0.7.8

Answer Not Provided

Answer of exercise 0.7.9

Question:

What is the value of $* + *5 + *6$?

⋆ Answer:

$$\begin{aligned} * + *5 + *6 &= *(1 \oplus 5 \oplus 6) \\ &= *(01_2 \oplus 101_2 \oplus 110_2) \\ &= *(010_2) \\ &= *2 \end{aligned}$$

Answer of exercise 0.7.10

Answer Not Provided

Answer of exercise 0.7.11

Question:

What is the value of $*10+*20+*25$?

⋆ Answer:

$$\begin{aligned} *10+*20+*25 &= *(10 \oplus 20 \oplus 25) \\ &= *(01010_2 \oplus 10100_2 \oplus 11001_2) \\ &= *(00111_2) \\ &= *(4+2+1) \\ &= *7 \end{aligned}$$

Answer of exercise 0.7.12

Answer Not Provided

Answer of exercise 0.7.13

Question:

What is the value of $\sum_{i=1}^{7} *i = * + *2 + *3 + *4 + *5 + *6 + *7$?

⋆ Answer:

$$\begin{aligned} \sum_{i=1}^{7} *i &= * + *2 + *3 + *4 + *5 + *6 + *7 \\ &= * + (*2 + *3) + (*4 + *5) + (*6 + *7) \\ &= * + (*) + (*) + (*) \\ &= 4 \times * \\ &= 0 \end{aligned}$$

Answer of exercise 0.7.14

Answer Not Provided

Answer of exercise 0.7.15

Question:

Using what you learned in exercises 0.7.13 and 0.7.14 , what is the value of $\sum_{i=1}^{4k+1} *i = * + *2 + *3 + \cdots + *(4k) + *(4k+1)$?

⋆ Answer:

$$
\begin{aligned}
\sum_{i=1}^{4k+1} *i &= * + *2 + *3 + \cdots + *(4k) + *(4k+1) \\
&= * + (*2 + *3) + \cdots + (*(4k) + *(4k+1)) \\
&= * + \overbrace{(*2 + *3) + \cdots + (*(4k) + *(4k+1))}^{4k} \\
&= * + \overbrace{(*) + \cdots + (*)}^{2k} \\
&= (2k+1) \times * \\
&= *
\end{aligned}
$$

Answer of exercise 0.7.16

Answer Not Provided

Answer of exercise 0.7.17

Answer Not Provided

Answer of exercise 0.7.18

Question:

Find an option of the game

\ | \   / \ / \ | | / | | | | / \ / | \

i.e. $(3,7,9)$, that is in $\mathcal{P}$.

★ Answer:

The game

/ | | \ | | / \ | \ | / / |

i.e. $(3,7,4)$, is in $\mathcal{P}$.

Answer of exercise 0.7.19

Answer Not Provided

Answer of exercise 0.7.20

Question:

Find an option of the game

/ \ | | \ /   / \ | \ \ \ /

i.e. $(6,2,5)$, that is in $\mathcal{P}$.

★ Answer:

The game

| | | | / / | / | \ | |

i.e. $(6,2,4)$, is in $\mathcal{P}$.

Answer of exercise 0.7.21

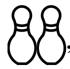
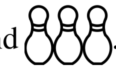
Answer Not Provided

Answer of exercise 0.7.22

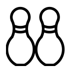
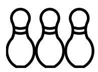
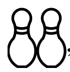
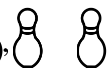
Answer Not Provided

Answer of exercise 0.7.23

Question:

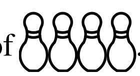
Find the nimber values of , , and .

★ Answer:

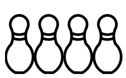
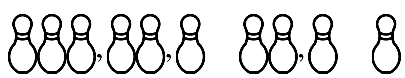
-  has a single move to zero, so  $=*$.
-  $= {}^*\{ 0,$  $\} = {}^*\{ 0, * \} = *\,2$
-  $= {}^*\{$ , ,  $\} = {}^*\{ *\,2, *, * + * \} = *\,3$

Answer of exercise 0.7.24

Question:

Find the nimber value of . You will likely need to use your answer to exercise 0.7.23 .

★ Answer:

 $= {}^*\{$  $\}$

$$= {}^*\{ *\,3, *\,2, * + *\,2, * + * \}$$
$$= {}^*\{ *\,3, *\,2, *\,3, 0 \}$$
$$= *$$

Answer of exercise 0.7.25

Answer Not Provided

Answer of exercise 0.7.26

Answer Not Provided

Answer of exercise 0.7.27

Answer Not Provided

Answer of exercise 0.7.28

Answer Not Provided

Answer of exercise 0.8.0

Question:

Complete the following truth table for $x_1 \wedge (x_2 \vee x_3)$.

x_1	x_2	x_3	$x_2 \vee x_3$	$x_1 \wedge (x_2 \vee x_3)$
F	F	F	F	F
F	F	T		
F	T	F		
F	T	T		
T	F	F		
T	F	T		
T	T	F		
T	T	T		

⋆ Answer:

x_1	x_2	x_3	$x_2 \vee x_3$	$x_1 \wedge (x_2 \vee x_3)$
F	F	F	F	F
F	F	T	T	F
F	T	F	T	F
F	T	T	T	F
T	F	F	F	F
T	F	T	T	T
T	T	F	T	T
T	T	T	T	T

Answer of exercise 0.8.1

Answer Not Provided

Answer of exercise 0.8.2

Question:

How many rows would we need in a truth table for a formula with four boolean variables?

⋆ Answer:

Twice as many rows as three-variables, so 16 rows.

Answer of exercise 0.8.3

Answer Not Provided

Answer of exercise 0.8.4

Question:

Consider AVOID TRUE played on the formula $(x_1 \vee x_2) \wedge x_3$. Assume the players are planning to play on variables in this order: x_1, x_2, then x_3. Evaluate the formula after each (attempted) move (and show your work). What is the first of these moves that can't be made because it will be illegal?

⋆ Answer:

- $x_1 \leftarrow \mathbf{T}$: $(\mathbf{T} \vee \mathbf{F}) \wedge \mathbf{F} = (\mathbf{T}) \wedge \mathbf{F} = \mathbf{F}$
- $x_2 \leftarrow \mathbf{T}$: $(\mathbf{T} \vee \mathbf{T}) \wedge \mathbf{F} = (\mathbf{T}) \wedge \mathbf{F} = \mathbf{F}$
- $x_3 \leftarrow \mathbf{T}$: $(\mathbf{T} \vee \mathbf{T}) \wedge \mathbf{T} = (\mathbf{T}) \wedge \mathbf{T} = \mathbf{T}$

The third move of flipping x_3 is illegal.

Answer of exercise 0.8.5

Answer Not Provided

Answer of exercise 0.8.6

Answer Not Provided

Answer of exercise 0.8.7

Question:

Consider AVOID TRUE played on the formula $(x_1 \vee x_2) \wedge (x_2 \vee x_3 \vee x_4)$. Which variables can the first player *not* choose?

⋆ Answer:

x_2 cannot be chosen because it is in all of the clauses.

Answer of exercise 0.8.8

Question:

Consider AVOID TRUE played on the formula $(x_1 \vee x_2) \wedge (x_2 \vee x_3 \vee x_4)$. What is the outcome class of this position? (This is a continuation of exercise 0.8.7 .)

⋆ Answer:

Choosing x_1 results in the position $(x_2 \vee x_3 \vee x_4)$. Since there is only one clause remaining and no extra false variables, there are no further moves and this position is in $\mathcal{P}$. That means the initial position is in $\mathcal{N}$.

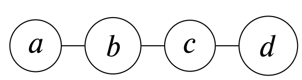
Answer of exercise 0.8.9

Answer Not Provided

Answer of exercise 0.8.10

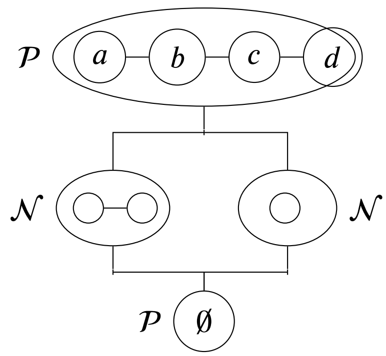
Question:

Use a game tree to determine the outcome class of NODE KAYLES on a path with four vertices.



Hint: combine moves that result in equivalent graphs into one option instead of separate.

⋆ Answer:



Answer of exercise 0.8.11

Answer Not Provided

Answer of exercise 0.8.12

Question:

Find the outcome class and winning strategy for CHOMP on any rectangular board of size $2 \times n, n \geq 2$.

⋆ Answer:

We know from the strategy stealing argument that this board is in $\mathcal{N}$. The first player can remove square $(2, n)$, leaving a board that is one square shorter in row 2 than in row 1. No matter what the second player does, the board can be returned to this state, until the other player is left with the single square $(1, 1)$.

Answer of exercise 0.8.13

Answer Not Provided

Answer of exercise 0.8.14

Question:

Determine the game value of the CHOMP positions $2 \times 2, 2 \times 3$, and 2×4.

⋆ Answer:

When $n = 2$ we get $*2$, when $n = 3$ we get $*4$, and when $n = 4$ we get $*5$.

Answer of exercise 0.8.15

Question:

Play $2 \times n$ CHOMP with someone else for a few different values of n and win every time.

⋆ Answer:

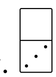
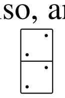
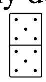
It's actually hard to not let on what the winning move is, especially if the other person asks to go first after learning the method. You may need to make some non-optimal moves the first couple rounds to throw them off!

Answer of exercise 0.8.16

Question:

Find all single-domino DOMINIM positions in $\mathcal{P}$.

⋆ Answer:

First, any domino with 0 on top and any number of pips on the bottom, e.g.  , is in $\mathcal{P}$. Also, any domino in which the top and bottom have the same number of pips, e.g.  ,  , etc. as this can be played as Tweedledum-Tweedledee. Because of this, any domino in which the top has fewer pips than the bottom is in $\mathcal{P}$, since every option is to a domino with a greater top than bottom, and this domino has as an option at least one of the previously noted $\mathcal{P}$-positions.

Answer of exercise 0.8.17

Answer Not Provided

Answer of exercise 0.9.0

Answer Not Provided

Answer of exercise 0.9.1

Question:

Find a closed formula for the sequence

$$-7, -2, 3, 8, 13, 18, \ldots$$

⋆ Answer:

The sequence is Δ^1-constant so it is arithmetic. $a_n = -7 + 5n$.

Answer of exercise 0.9.2

Answer Not Provided

Answer of exercise 0.9.3

Question:

Find a closed formula for the sequence

$$1, 0, 3, 10, 21, 36, 55, \ldots$$

⋆ Answer:

This sequence is Δ^2-constant so has a closed formula of the form $a_n = An^2 + Bn + C$. Solving for A, B, and C yields $a_n = 2n^2 - 7n + 6$.

Answer of exercise 0.9.4

Answer Not Provided

Answer of exercise 0.9.5

Question:

Find a closed formula for the sequence

$$4, 2, 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$$

⋆ Answer:

This is a geometric sequence with ratio $r = \frac{1}{2}$, so $a_n = 4 \cdot (\frac{1}{2})^n$

Answer of exercise 0.9.6

Answer Not Provided

Answer of exercise 0.9.7

Question:

Find a closed formula for the recurrence relation

$$a_n = \begin{cases} 2 & \text{if } n = 0 \\ 3 & \text{if } n = 1 \\ 2a_{n-1} - a_{n-2} & \text{otherwise} \end{cases}$$

⋆ Answer:

This relation has characteristic equation $x^2 - 2x + 1 = 0$, which simplifies to $(x-1)^2 = 0$. Since it has one repeated root we let

$$a_n = A_1(-1)^n + A_2 n(-1)^n$$

Plugging in the first two terms yields

$$2 = A_1, 3 = -A_1 - A_2$$

so $A_1 = 2, A_2 = -5$, and hence

$$a_n = 2(-1)^n - 5n(-1)^n$$

Answer of exercise 0.9.8

Answer Not Provided

Answer of exercise 0.9.9

Answer Not Provided

Answer of exercise 0.9.10

Question:

List the first 8 terms in the Beatty sequence $\mathcal{B}_\pi$.

⋆ Answer:

$3, 6, 9, 12, 15, 18, 21, 25$

Answer of exercise 0.9.11

Answer Not Provided

Answer of exercise 0.9.12

Question:

Consider the Beatty sequence

$$\mathcal{B}_{\sqrt{3}} = 1, 3, 5, 6, 8, 10, 12, 13, \ldots$$

Find the first 5 terms of its complementary sequence $\mathcal{B}_s$, and then determine the appropriate value for s.

⋆ Answer:

$\mathcal{B}_s = 2, 4, 7, 9, 11, \ldots$ with $s = \frac{\sqrt{3}}{\sqrt{3}-1}$.

Answer of exercise 0.9.13

Answer Not Provided

Answer of exercise 0.9.14

Answer Not Provided

Answer of exercise 0.9.15

Question:

Come up with a sequence such that successive terms have ratios that themselves have a fixed ratio between them (i.e. like a Δ^2-constant sequence but with ratios instead of differences).

⋆ Answer:

Consider $3, 3, 6, 24, 192, 3072, 98304 \ldots$.

The ratios are $\cdot 1, \cdot 2, \cdot 4, \cdot 8, \cdot 16, \cdot 32, \ldots$, which themselves have a constant ration of $\cdot 2$.

Answer of exercise 1.1.0

Question:

Find the shortest possible game of UNDIRECTED VERTEX GEOGRAPHY in the graph G above.

⋆ Answer:

The shortest game on G has 5 vertices: begin at the apex and visit each vertex in the outer pentagon.

Answer of exercise 1.1.1

Answer Not Provided

Answer of exercise 1.1.2

Answer Not Provided

Answer of exercise 1.1.3

Question:

Find the shortest possible game of UNDIRECTED VERTEX GEOGRAPHY in the graph J above.

⋆ Answer:

The shortest game in J has one move: Along the pendant edge to the sole vertex of degree one.

Answer of exercise 1.1.4

Answer Not Provided

Answer of exercise 1.1.5

Question:

Is the Petersen Graph hamiltonian? Is it traceable?

⋆ Answer:

The Petersen Graph is traceable (a path starting in the outer cycle can include all vertices, then move to the inner cycle and include the remaining vertices). It is not hamiltonian.

Proof. We proceed by contradiction. Assume the graph P is hamiltonian. Since it has 10 vertices, there is a cycle C_{10} on 10 vertices, which also contains 10 edges. Since P has 15 edges there are 5 remaining edges joining vertices in the 10-cycle. There is no way for this to occur without creating at least one 3- or

4-cycle, of which P has none. Therefore P does not contain a 10-cycle and is not hamiltonian. □

P is the smallest 3-regular graph without a bridge that is not hamiltonian.

Answer of exercise 1.1.6

Answer Not Provided

Answer of exercise 1.1.7

Answer Not Provided

Answer of exercise 1.1.8

Question:

Prove that every tree is bipartite.

⋆ Answer:

There are a number of ways to prove this. We will do so by finding a bipartition of the vertices.

Proof. Let T be a tree and v any vertex in T. By definition, T is connected and has no cycles. For every vertex u in T, place u into the set E if its distance from v is even, and into set O if its distance from v is odd. This creates a bipartition of T since no two vertices in the same set are connected, otherwise T would contain a cycle. Therefore, T is bipartite. □

Answer of exercise 1.1.9

Answer Not Provided

Answer of exercise 1.1.10

Question:

Let C_n be the cycle graph on n vertices (drawn as an n-gon). Find a formula for the size $L(C_n)$ of the largest possible matching in C_n, and then find a formula for the size $S(C_n)$ of the smallest possible maximal matching in C_n, in terms of n.

⋆ Answer:

If n is even then $L(C_n) = \frac{n}{2}$, and if n is odd then $L(C_n) = \frac{n-1}{2}$. So $L(C_n) = \lfloor \frac{n}{2} \rfloor$.
For the smallest maximal matching, we can choose every third edge. Thus, $S(C_n) = \lceil \frac{n}{3} \rceil$.

Answer of exercise 1.1.11

Answer Not Provided

Answer of exercise 1.2.0

Question:

What, if any, are the restrictions on n and m such that $K_{m,n}$ has (i) an euler circuit and (ii) an euler tour?

⋆ Answer:

For an euler circuit, we need all even degrees. So m and n both much be even. For an euler tour, we need exactly two odd degree vertices. The only way this is possible in a complete bipartite graph is if one part is equal to 2 and the other is odd.

Answer of exercise 1.2.1

Answer Not Provided

Answer of exercise 1.2.2

Answer Not Provided

Answer of exercise 1.3.0

Question:

Prove that if a digraph is strongly connected then it is also weakly connected.

⋆ Answer:

Proof. Let u, v be vertices in G. If G is strongly connected then there is, by definition, a $u - v$ path in G. Hence G is also weakly connected. □

Answer of exercise 1.3.1

Answer Not Provided

Answer of exercise 1.3.2

Answer Not Provided

Answer of exercise 1.3.3

Answer Not Provided

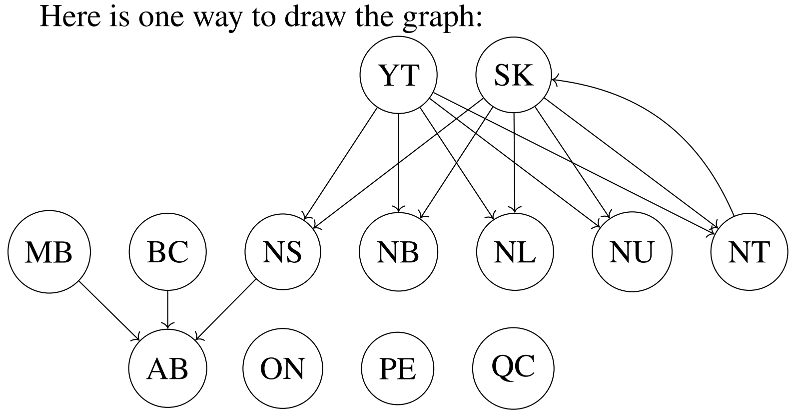
Answer of exercise 1.3.4

Question:

Consider the game of GEOGRAPHY played on Canadian Provinces and Territories. Draw the corresponding DIRECTED GEOGRAPHY graph. Label your vertices with the two-character shorthands. You don't know what those are, eh? Here's the list: Alberta (AB), British Columbia (BC), Manitoba (MB), New Brunswick (NB), Newfoundland and Labrador (NL), Nova Scota (NS), Northwest Territories (NT), Nunavut (NU), Ontario (ON), Prince Edward Island (PE), Quebec (QC), Saskatchewan (SK), Yukon (YT).

⋆ Answer:

Here is one way to draw the graph:



Answer of exercise 1.3.5

Answer Not Provided

Answer of exercise 1.3.6

Answer Not Provided

Answer of exercise 1.3.7

Answer Not Provided

Answer of exercise 2.1.0

Question:

Rewrite $^*\left\{\ 0, *, *5, *7\ \right\}$ in option notation. Do not simplify to a single nimber value.

★ Answer:

$\left\{\ 0, *, *5, *7 \mid 0, *, *5, *7\ \right\}$

Answer of exercise 2.1.1

Answer Not Provided

Answer of exercise 2.1.2

Question:

Simplify $\left\{\ 0, *, *2, *4 \mid 0, *, *2, *4\ \right\}$ to a single nimber.

★ Answer:

$*3$

Answer of exercise 2.1.3

Answer Not Provided

Answer of exercise 2.1.4

Question:

Can we simplify $\left\{\ 0, *, *2, *3 \mid 0, *, *2\ \right\}$ to a single nimber? If so, provide that nimber value.

★ Answer:

No, $\left\{\ 0, *, *2, *3 \mid 0, *, *2\ \right\}$ does not simplify.

Answer of exercise 2.1.5

Question:

Can we simplify $\left\{\ 0, *, *2, *4 \mid 0, *, *2, *5, *6\ \right\}$ to a single nimber? If so, provide that nimber value.

⋆ Answer:

Yes, it simplifies to $*3$.

Answer of exercise 2.1.6

Answer Not Provided

Answer of exercise 2.1.7

Answer Not Provided

Answer of exercise 2.1.8

Question:

Can we simplify $\left\{ *, *2, *4, *5 \,\middle|\, *2, *4, \{ 0 \,|\, 0 \} \right\}$ to a single nimber? If so, provide that nimber value. Show your work.

⋆ Answer:

$\{ 0 \,|\, 0 \} = *$, so the position simplifies to $\{ *, *2, *4, *5 \,|\, *2, *4, * \}$, which equals 0.

Answer of exercise 2.1.9

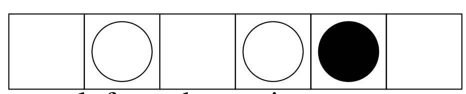
Answer Not Provided

Answer of exercise 2.1.10

Answer Not Provided

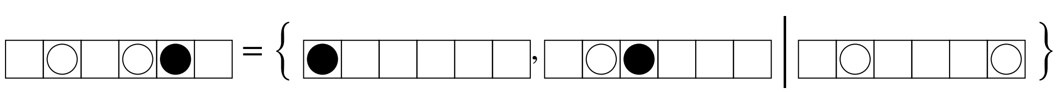
Answer of exercise 2.1.11

Question:

What is the value of the KONANE position 

? You can simplify your analysis by using the result from the section text.

⋆ Answer:



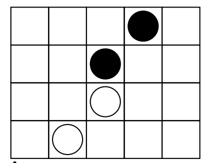
$= \{ 0, * \,|\, 0 \}$

Answer of exercise 2.1.12

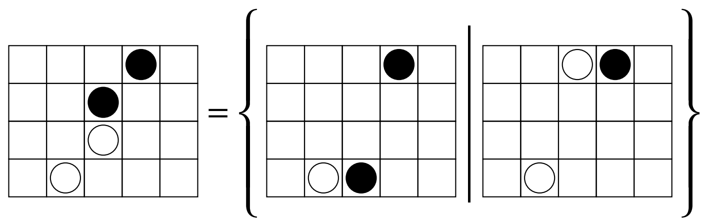
Answer Not Provided

Answer of exercise 2.1.13

Question:

What is the value of the KONANE position  ? You can simplify your analysis by using the result from the section text.

★ Answer:



$$= \{ * \mid * \}$$
$$= 0$$

Answer of exercise 2.1.14

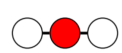
Answer Not Provided

Answer of exercise 2.1.15

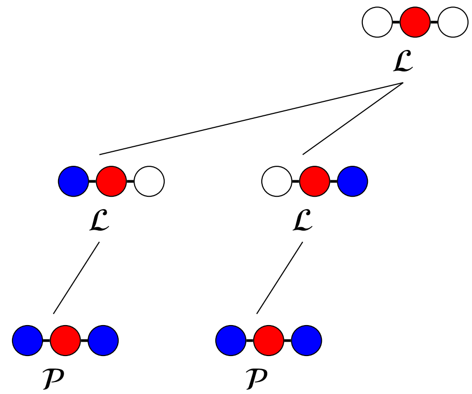
Answer Not Provided

Answer of exercise 2.2.0

Question:

Draw out the entire game tree for  and label with outcome classes to show that it is in $\mathcal{L}$. Make sure your edges are pointing in the right direction!

★ Answer:



Answer of exercise 2.2.1

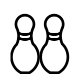
Answer Not Provided

Answer of exercise 2.2.2

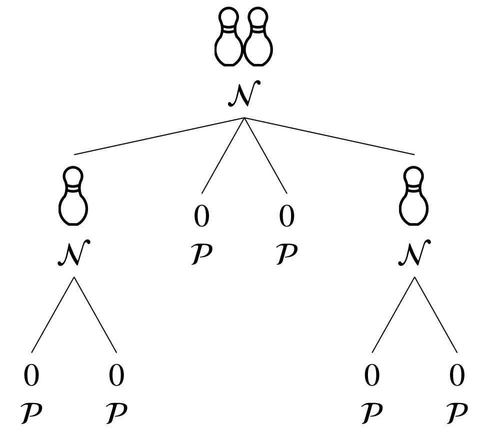
Answer Not Provided

Answer of exercise 2.2.3

Question:

Draw out the entire (partisan) game tree for  . Label your tree with the outcome classes. Do you get the correct outcome class that you would using an impartial game tree?

⋆ Answer:

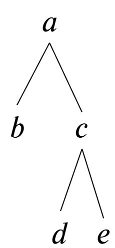


Yes, we get the same outcome class ($\mathcal{N}$) that we would using an impartial tree.

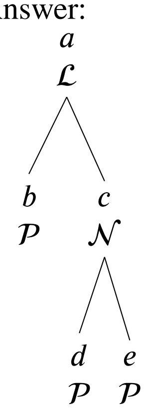
Answer of exercise 2.2.4

Question:

Label all positions of the following game tree with their outcome classes.



⋆ Answer:



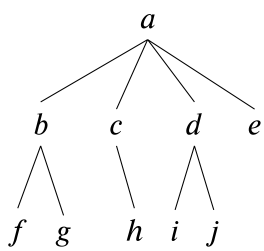
Answer of exercise 2.2.5

Answer Not Provided

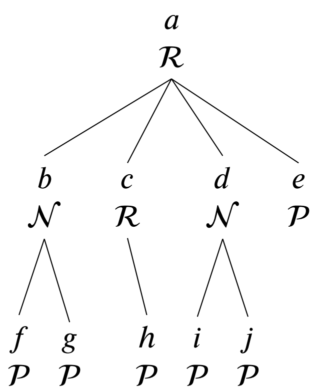
Answer of exercise 2.2.6

Question:

Label all positions of the following game tree with their outcome classes.



⋆ Answer:



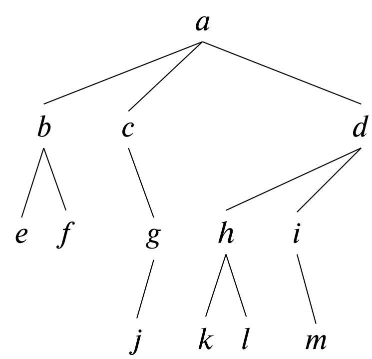
Answer of exercise 2.2.7

Answer Not Provided

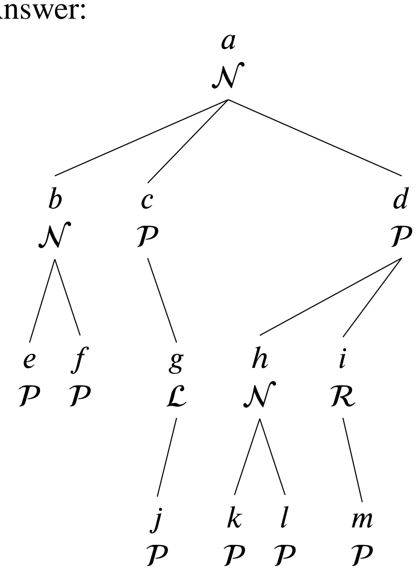
Answer of exercise 2.2.8

Question:

Label all positions of the following game tree with their outcome classes.



★ Answer:



Answer of exercise 2.2.9

Answer Not Provided

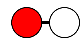
Answer of exercise 2.2.10

Question:

What is the smallest game tree you can draw that has an outcome class of $\mathcal{R}$? Can you find a COL position that has that tree?

★ Answer:

Here is a tree that has only two nodes (positions):

A simple COL position with this tree is:  .

Answer of exercise 2.2.11

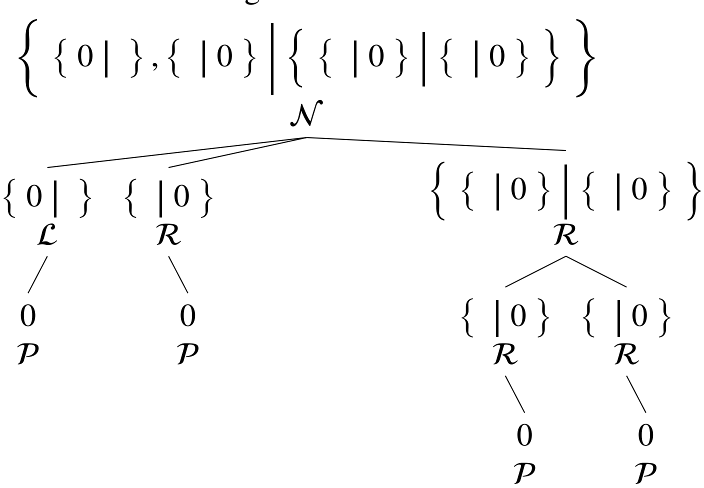
Question:

Translate this position into a game tree and determine the outcome class:

$$\left\{ \{0|\ \},\{\ |0\} \middle| \left\{ \{\ |0\} \middle| \{\ |0\} \right\} \right\}.$$

⋆ Answer:

Here is the labelled game tree:



The root outcome class is $\mathcal{N}$.

Answer of exercise 2.2.12

Answer Not Provided

Answer of exercise 2.2.13

Answer Not Provided

Answer of exercise 2.2.14

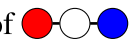
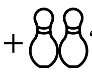
Answer Not Provided

Answer of exercise 2.3.0

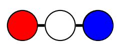
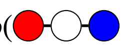
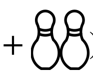
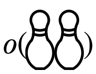
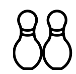
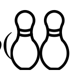
Answer Not Provided

Answer of exercise 2.3.1

Question:

What is the outcome class of  + ?

⋆ Answer:

 $= 0$. So $o($ + $) = o($$)$.  $= {}^{*}\{\,0, *\,\} = *2$, so $o($ $= \mathcal{N}$. The overall outcome class is also $\mathcal{N}$.

Answer of exercise 2.3.2

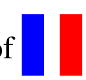
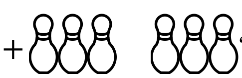
Answer Not Provided

Answer of exercise 2.3.3

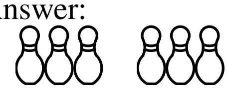
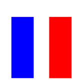
Answer Not Provided

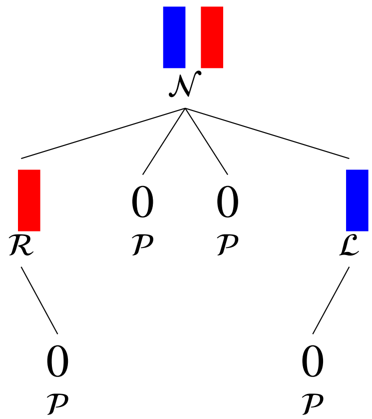
Answer of exercise 2.3.4

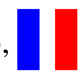
Question:

What is the outcome class of  + ?

⋆ Answer:

 is in $\mathcal{P}$, by Tweedledum-Tweedledee. To find the sum's outcome class, we can draw out the game tree for :



Since both L and R have winning moves (both to 0),  $\in \mathcal{N}$, meaning that the overall game is also in $\mathcal{N}$.

Answer of exercise 2.3.5

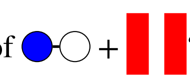
Answer Not Provided

Answer of exercise 2.3.6

Answer Not Provided

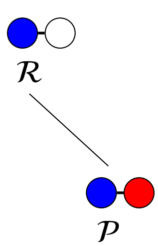
Answer of exercise 2.3.7

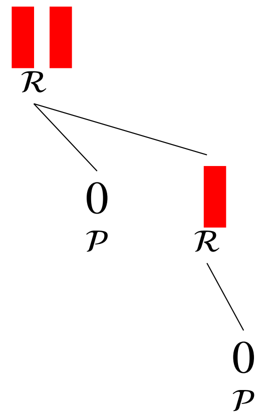
Question:

What is the outcome class of ?

⋆ Answer:

We start by finding the outcome class of each component using trimmed game trees:





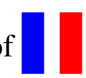
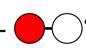
Both components are in $\mathcal{R}$, so, by the table in the section, the sum must also be in $\mathcal{R}$.

Answer of exercise 2.3.8

Answer Not Provided

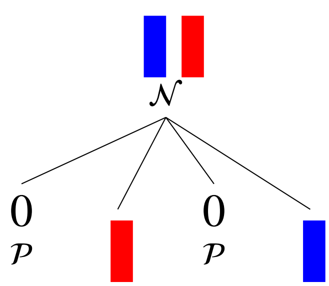
Answer of exercise 2.3.9

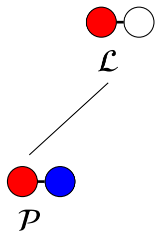
Question:

What is the outcome class of  + ?

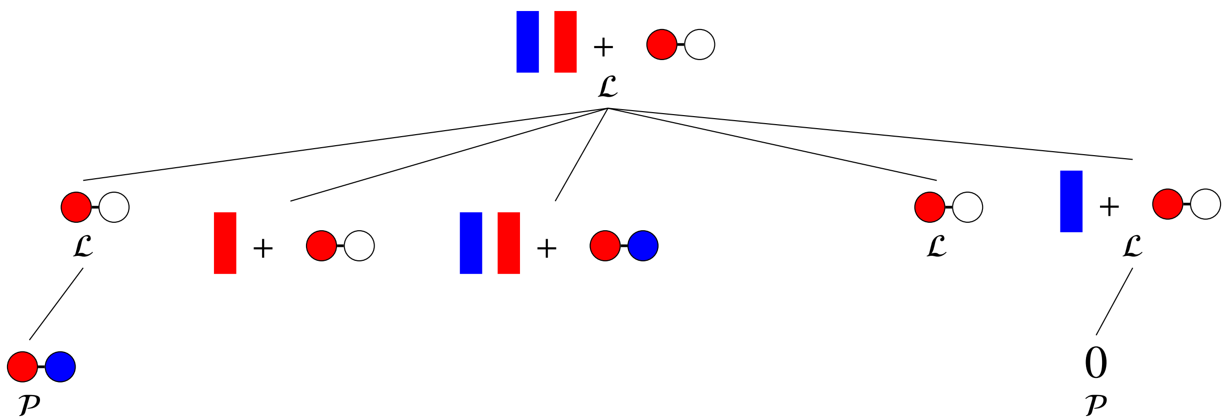
⋆ Answer:

We start by finding the outcome class of each component using trimmed game trees:





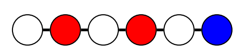
Unfortunately, since the components are in $\mathcal{L}$ and $\mathcal{N}$, we need to look at the tree of the sum.



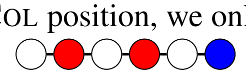
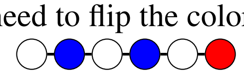
Thus the sum is in $\mathcal{L}$.

Answer of exercise 2.4.0

Question:

Find the COL position that is the (obvious) negation of  . (You don't need to show work if you use the (super fast) trick given in the chapter.)

⋆ Answer:

To negate a COL position, we only need to flip the color of each token. Thus, the negation of  is  .

Answer of exercise 2.4.1

Answer Not Provided

Answer of exercise 2.4.2

Question:

Find the TOPPLING DOMINOES position that is the (obvious) negation of  . (You don't need to show work if you use the (super fast) trick given in the chapter.)

⋆ Answer:

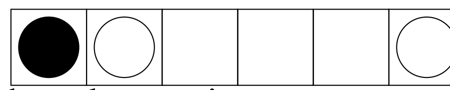
To negate a TOPPLING DOMINOES position, we only need to flip the color of each domino. Thus, the negation of  is  .

Answer of exercise 2.4.3

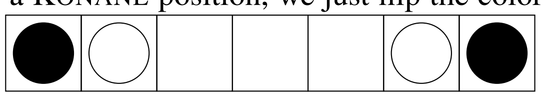
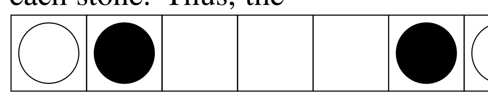
Answer Not Provided

Answer of exercise 2.4.4

Question:

Find the KONANE position that is the (obvious) negation of  . (Hint: KONANE positions can be negated just like the other rulesets we've seen.)

⋆ Answer:

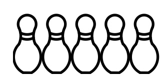
To negate a KONANE position, we just flip the color of each stone. Thus, the negation of  is  .

Answer of exercise 2.4.5

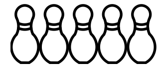
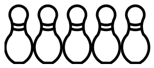
Answer Not Provided

Answer of exercise 2.4.6

Question:

Find the KAYLES position that is the (obvious) negation of  . (Hint: what is the negation of a position from an impartial ruleset?)

⋆ Answer:

Every impartial game is it's own negative, so the negation of  is simply  .

Answer of exercise 2.4.7

Answer Not Provided

Answer of exercise 2.4.8

Question:

Simplify $G = -\{\, 0, *2 \mid 0, *2 \,\}$ by removing the minus sign and reducing your answer as much as you know how.

⋆ Answer:

$$
\begin{aligned}
-\{\, 0, *2 \mid 0, *2 \,\} &= -\, * && (2.1)\\
&= * && (2.2)
\end{aligned}
$$

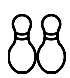
The last step occurs because every nimber is equal to its own negative.

Answer of exercise 2.4.9

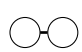
Answer Not Provided

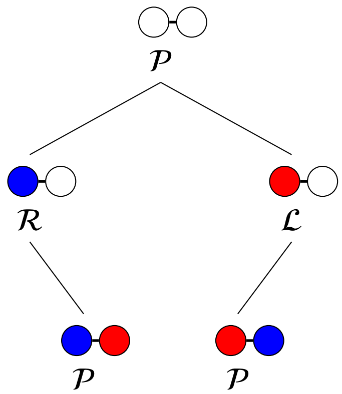
Answer of exercise 2.4.10

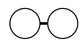
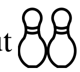
Question:

Is COL position  equal to  ? Prove your answer. (Yes, this is from one of the team exercises.)

⋆ Answer:

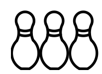
First we find the outcome class of  :



They are not equal because  $= 0$, but  $\in \mathcal{N}$.

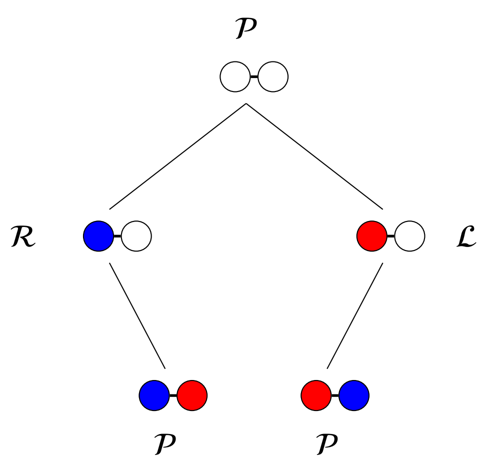
Answer of exercise 2.4.11

Question:

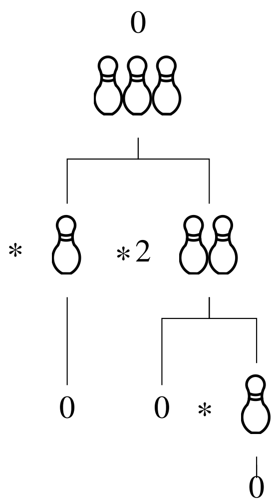
Is COL position  equal to  ? Prove your answer.

⋆ Answer:

Let's check the outcome class of these two positions. First the COL position:



Next the KAYLES position:



Both positions are in $\mathcal{P}$, so they are equal. (Both are equal to 0.)

Answer of exercise 2.4.12

Question:

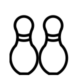
Is COL position ○ equal to  ? Prove your answer.

★ Answer:

○ = { 🔵 | 🔴 } = $\{ 0 \mid 0 \} = *$.  = $^*\{ 0 \} = *$, so these are equal!

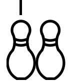
Answer of exercise 2.4.13

Question:

Is COL position ○ equal to  ? Prove your answer.

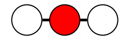
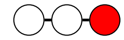
★ Answer:

○ = { 🔵 | 🔴 } = $\{ 0 \mid 0 \} = *$.

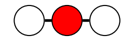
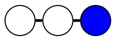
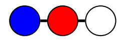
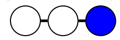
However,  = $^*\{$ 0,  $\}$ = $^*\{ 0, * \} = *2$. $* \neq *2$, so these are not equal.

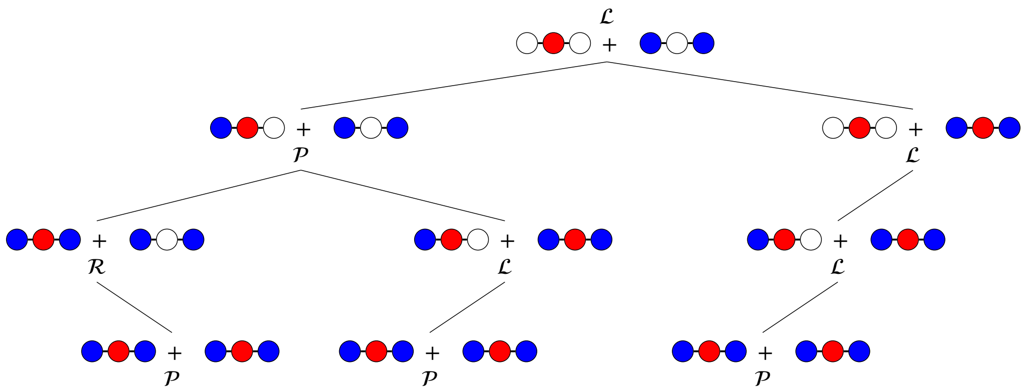
Answer of exercise 2.4.14

Question:

From another of the team exercises, is  equal to ? (Both are COL.) Prove your answer.

⋆ Answer:

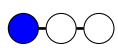
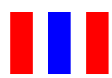
To determine whether these are equal, we will check whether  +  $\in \mathcal{P}$. The entire tree is a bit too big to analyze, so let's (carefully) check one of the left options first,  + :



Since the sum position has a winning move for L, it is not in $\mathcal{P}$, so the initial positions are not equivalent.

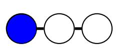
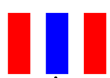
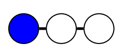
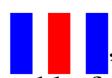
Answer of exercise 2.4.15

Question:

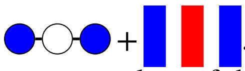
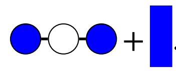
Is COL position  equal to ? Prove your answer.

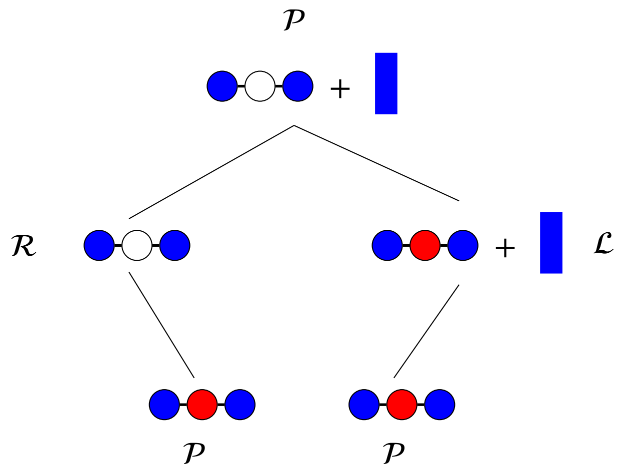
⋆ Answer:

These two *are* equal! (If we check the outcome classes, we will see that they are both in $\mathcal{R}$.) We can prove this by showing that the difference between them is 0.

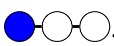
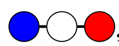
G =  −  =  + , so we'll analyze that sum. Unfortunately, the tree is too big to comfortably fit here, so we'll have to look at the options individually. We just need to show that none of the options is a winning move. That means we need to show that each of the six options has a winning response from the other player.

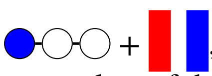
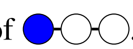
First the left-options:

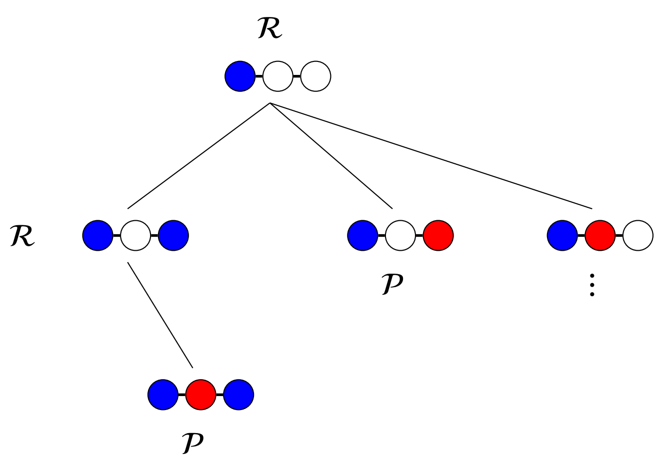
- . This has a right-option to: . Let's find the outcome class of that.



  The outcome class of that right-option is $\mathcal{P}$, so that's a winning response.

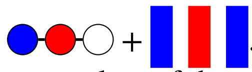
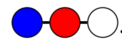
- . This has a right-option to , which is 0, a winning response.

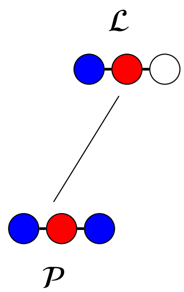
- , which has a right-option of . Let's find the outcome class of that.



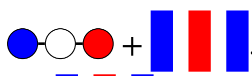
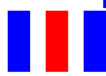
The outcome class of this right-option is $\mathcal{R}$, so it's a winning response.

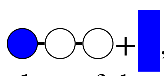
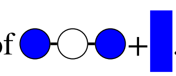
None of those left options are a winning. Let's look at the right-options now:

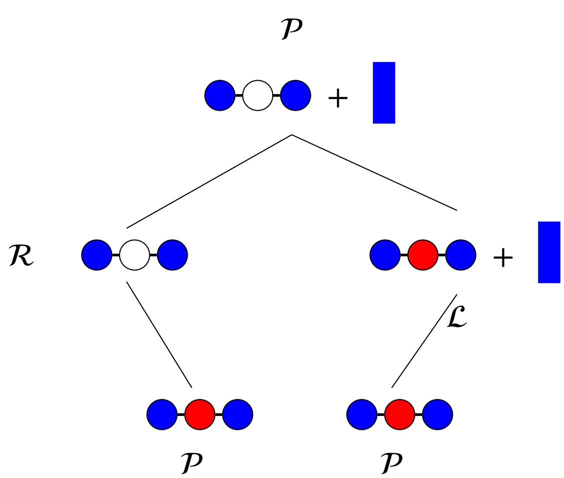
- . This has a left-option to: . Let's find the outcome class of that.



The outcome class of that left-option is $\mathcal{L}$, so that's a winning response.

- . The COL position is zero, so the whole thing is equal to . This has a left-option to 0, a winning response.

- , which has a left-option of . Let's find the outcome class of that.



The outcome class of this left-option is $\mathcal{P}$, so it's a winning response.

None of the options of G are winning options, so $G = 0$. This means that the original two positions are equal!

Answer of exercise 2.4.16

Answer Not Provided

Answer of exercise 2.4.17

Answer Not Provided

Answer of exercise 2.4.18

Answer Not Provided

Answer of exercise 2.4.19

Answer Not Provided

Answer of exercise 2.4.20

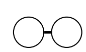
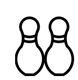
Answer Not Provided

Answer of exercise 2.4.21

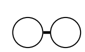
Answer Not Provided

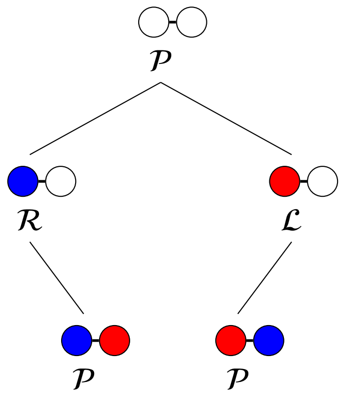
Answer of exercise 2.5.0

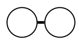
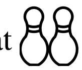
Question:

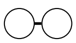
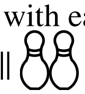
What is the inequality relationship between Col position  and  ? Prove your answer. (This is a follow-up of exercise 2.4.10.)

⋆ Answer:

First we're going to find the outcome class of  :

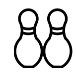


So  $\in \mathcal{P}$. We know already that  $= {}^{*}\{\, 0, * \,\} \in \mathcal{N}$, so they are confused with each other:

 $\|$ 

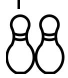
Answer of exercise 2.5.1

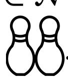
Question:

What is the inequality relationship between COL position  and  ? Prove your answer. (This is a follow-up of exercise 2.4.13.)

★ Answer:

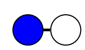
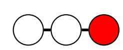
 $= \{$  $|$  $\} = \{\, 0 \mid 0 \,\} = *$.

However,  $= {}^{*}\{\, 0,$  $\} = {}^{*}\{\, 0, * \,\} = *2$. Thus, since $* - *2 = * + *2 = *3 \in \mathcal{N}$, these games are confused with each other.

 $\|$ .

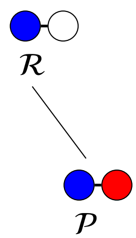
Answer of exercise 2.5.2

Question:

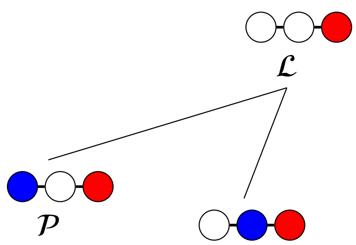
What is the inequality relationship between COL positions  and  ? Prove your answer.

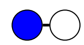
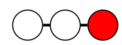
★ Answer:

We can solve this by looking at the two outcome classes.



and



Since the first is in $\mathcal{R}$ and the second is in $\mathcal{L}$,  $<$  .

Answer of exercise 2.5.3

Answer Not Provided

Answer of exercise 2.5.4

Answer Not Provided

Answer of exercise 2.5.5

Answer Not Provided

Answer of exercise 2.5.6

Answer Not Provided

Answer of exercise 2.5.7

Question:

$G \in \mathcal{L} \cup \mathcal{N}$. Rewrite this as an equivalent comparison of G and 0.

$\star$ Answer:

G is either greater than zero or confused with zero, so $G \mathrel{|\!\triangleright} 0$

Answer of exercise 2.5.8

Answer Not Provided

Answer of exercise 2.5.9

Answer Not Provided

Answer of exercise 2.5.10

Question:

$G \ngeq 0$. Rewrite this as an equivalent comparison of G and 0 using a different symbol.

⋆ Answer:

$G \mathrel{\triangleleft\!|} 0$

Answer of exercise 2.5.11

Answer Not Provided

Answer of exercise 2.5.12

Question:

$G \leq 0$. Write the equivalent expression of G as an element of the union of outcome classes.

⋆ Answer:

$G \in \mathcal{R} \cup \mathcal{P}$.

Answer of exercise 2.5.13

Answer Not Provided

Answer of exercise 2.5.14

Answer Not Provided

Answer of exercise 2.5.15

Answer Not Provided

Answer of exercise 2.5.16

Question:

Is the relation “has the same decimal part as” on the set of all reals reflexive, symmetric, and/or transitive? Demonstrate.

⋆ Answer:

Reflexive: Every real number has the same decimal part as itself.

Symmetric: If x has the same decimal part as y then y has the same decimal part as x.

Transitive: If x and y have the same decimal parts, and y and z have the same decimal parts, then x and z have the same decimal parts.

Answer of exercise 2.5.17

Answer Not Provided

Answer of exercise 2.5.18

Question:

Is the relation “has the same remainder when divided by 3” on the set of all naturals reflexive, symmetric, and/or transitive? Demonstrate.

⋆ Answer:

Reflexive: xRx is clear

Symmetric: If $x/3$ has the same remainder as $y/3$, then $y/3$ has the same remainder as $x/3$.

Transitive: If $x/3$ has the same remainder as $y/3$, and $y/3$ has the same remainder as $z/3$, then $x/3$ and $z/3$ have the same remainder.

Answer of exercise 2.5.19

Answer Not Provided

Answer of exercise 2.5.20

Answer Not Provided

Answer of exercise 2.6.0

Question:

$G = \left\{ \left\{ \,\middle|\, 0 \right\}, 0 \,\middle|\, 0 \right\}$. Determine whether any of G's options are dominated. If they are, remove them and simplify G as much as you know how. Show all your work.

⋆ Answer:

$\left\{ \,\middle|\, 0 \right\} - 0 = \left\{ \,\middle|\, 0 \right\} \in \mathcal{R}$, so $\left\{ \,\middle|\, 0 \right\} < 0$. That means that 0 dominates $\left\{ \,\middle|\, 0 \right\}$ for L. Thus, $G = \left\{ 0 \,\middle|\, 0 \right\} = *$.

Answer of exercise 2.6.1

Answer Not Provided

Answer of exercise 2.6.2

Question:

$G = \left\{ \left\{ 0 \,\middle|\, \right\}, 0 \,\middle|\, \left\{ 0 \,\middle|\, \right\}, 0 \right\}$. Determine whether any of G's options are dominated. If they are, remove them and simplify G as much as you know how. Show all your work.

⋆ Answer:

For L, we compare the two options: $\left\{ 0 \,\middle|\, \right\} - 0 = \left\{ 0 \,\middle|\, \right\} \in \mathcal{L}$, so $\left\{ 0 \,\middle|\, \right\} > 0$. That means that 0 is dominated by $\left\{ 0 \,\middle|\, \right\}$ for L. For R, we compare the options: $\left\{ 0 \,\middle|\, \right\} - 0 = \left\{ 0 \,\middle|\, \right\} \in \mathcal{L}$. Thus, $\left\{ \,\middle|\, 0 \right\}$ is dominated by 0 for R. Thus, $G = \left\{ \left\{ 0 \,\middle|\, \right\} \,\middle|\, 0 \right\}$.

Answer of exercise 2.6.3

Answer Not Provided

Answer of exercise 2.6.4

Question:

$G = \left\{ *, 0 \,\middle|\, 0, * \right\}$. Determine whether any of G's options are dominated. If they are, remove them and simplify G as much as you know how. Show all your work.

⋆ Answer:

Since $* \in \mathcal{N}$, $* \parallel 0$. We can't simplify either of these for either player. This works out with what we already know, because $G = {}^{*}\left\{ 0, * \right\} = *2$.

Answer of exercise 2.6.5

Answer Not Provided

Answer of exercise 2.6.6

Answer Not Provided

Answer of exercise 2.6.7

Answer Not Provided

Answer of exercise 3.1.0

Question:

Find the integer value of  +  + .

⋆ Answer:

 +  +  $= 1 + 1 + 1 = 3$

Answer of exercise 3.1.1

Answer Not Provided

Answer of exercise 3.1.2

Answer Not Provided

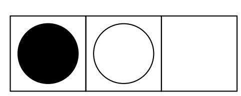
Answer of exercise 3.1.3

Answer Not Provided

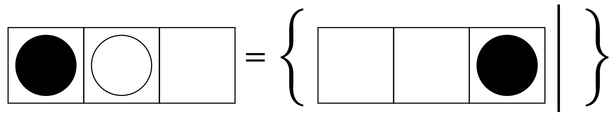
Answer of exercise 3.1.4

Question:

What is the value of this KONANE position?



⋆ Answer:



$= 1$

Answer of exercise 3.1.5

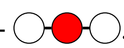
Answer Not Provided

Answer of exercise 3.1.6

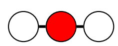
Answer Not Provided

Answer of exercise 3.1.7

Question:

Find the integer value of  + .

⋆ Answer:

 +  $= -2 + 2 = 0$

Answer of exercise 3.1.8

Answer Not Provided

Answer of exercise 3.1.9

Question:

Simplify $-1 + \ast 3 + 4 + \ast$.

⋆ Answer:

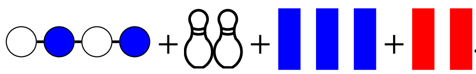
$-1 + \ast 3 + 4 + \ast = -1 + 4 + \ast 3 + \ast = -3 + \ast 2$

Answer of exercise 3.1.10

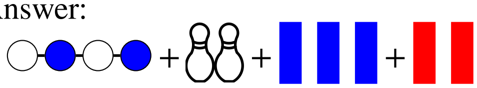
Answer Not Provided

Answer of exercise 3.1.11

Question:

Find the value of .

⋆ Answer:

 $= -2 + *2 + 3 + -2 = -1 + *2$

Answer of exercise 3.1.12

Answer Not Provided

Answer of exercise 3.1.13

Question:

Rewrite $\{\, 1 \mid 4 \,\} - 3$ as a single position and use basic arithmetic to simplify its options.

⋆ Answer:

$$\begin{aligned}\{\, 1 \mid 4 \,\} - 3 &= \left\{\, 1 - 3 \;\middle|\; 4 - 3, \{\, 1 \mid 4 \,\} - 2 \,\right\} \\ &= \left\{\, -2 \;\middle|\; 1, \{\, 1 \mid 4 \,\} - 2 \,\right\}\end{aligned}$$

Answer of exercise 3.1.14

Question:

Rewrite $\{\, 2 \mid -2 \,\} - 1$ as a single position and use basic arithmetic to simplify its options. Use this to determine the position's outcome class.

⋆ Answer:

$$\begin{aligned}\{\, 2 \mid -2 \,\} - 1 &= \left\{\, 2 - 1 \;\middle|\; -2 - 1, \{\, 2 \mid -2 \,\} - 0 \,\right\} \\ &= \left\{\, 1 \;\middle|\; -3, \{\, 2 \mid -2 \,\} \,\right\}\end{aligned}$$

Both players have a winning option, so this game is in $\mathcal{N}$.

Answer of exercise 3.1.15

Answer Not Provided

Answer of exercise 3.1.16

Answer Not Provided

Answer of exercise 3.1.17

Answer Not Provided

Answer of exercise 3.1.18

Question:

Continue coding `evaluate_position` by implementing the bottom case: when there are no right or left options.

⋆ Answer:

There are no options, so the value is simply zero:

```
elif len(left_options) == 0 and len(right_options) == 0:
  return 0
```

Answer of exercise 3.1.19

Answer Not Provided

Answer of exercise 3.2.0

Question:

What is the single-number value of $\{ 3 \mid 7 \}$?

⋆ Answer:

$\{ 3 \mid 7 \} = 4$, by the simplest number theorem.

Answer of exercise 3.2.1

Answer Not Provided

Answer of exercise 3.2.2

Question:

What is the single-number value of $\{ 33 \mid 133 \}$?

⋆ Answer:

$\left\{ \, 33 \mid 133 \, \right\} = 34$, by the simplest number theorem.

Answer of exercise 3.2.3

Answer Not Provided

Answer of exercise 3.2.4

Question:

List three dyadic rationals between $\frac{1}{8}$ and $\frac{7}{8}$.

⋆ Answer:

There are many answers, but these are the three simplest dyadic rationals in the interval: $\frac{1}{4}, \frac{1}{2}, \frac{3}{4}$.

Answer of exercise 3.2.5

Answer Not Provided

Answer of exercise 3.2.6

Question:

What is the single-number value of $\left\{ \, 10 \mid 11 \, \right\}$?

⋆ Answer:

$\left\{ \, 10 \mid 11 \, \right\} = 10\frac{1}{2}$, by the simplest number theorem.

Answer of exercise 3.2.7

Answer Not Provided

Answer of exercise 3.2.8

Question:

What is the single-number value of $\left\{ \, -5\frac{1}{2} \;\middle|\; -5 \, \right\}$?

⋆ Answer:

$\left\{ \, -5\frac{1}{2} \;\middle|\; -5 \, \right\} = -5\frac{1}{4}$, by the simplest number theorem.

Answer of exercise 3.2.9

Answer Not Provided

Answer of exercise 3.2.10

Question:

What is the single-number value of $\left\{ -\frac{1}{2} \,\middle|\, \frac{1}{8} \right\}$?

★ Answer:

$\left\{ -\frac{1}{2} \,\middle|\, \frac{1}{8} \right\} = 0$, by the simplest number theorem.

Answer of exercise 3.2.11

Answer Not Provided

Answer of exercise 3.2.12

Question:

Find an x and y such that $\{ x \mid y \} = 9\frac{5}{8}$.

★ Answer:

$\left\{ 9\frac{1}{2} \,\middle|\, 9\frac{3}{4} \right\} = 9\frac{5}{8}$, by the simplest number theorem.

Answer of exercise 3.2.13

Answer Not Provided

Answer of exercise 3.2.14

Question:

Find an x and y such that $\{ x \mid y \} = -5\frac{11}{16}$.

★ Answer:

$\left\{ -5\frac{3}{4} \,\middle|\, -5\frac{5}{8} \right\} = -5\frac{11}{16}$, by the simplest number theorem.

Answer of exercise 3.2.15

Answer Not Provided

Answer of exercise 3.2.16

Question:

What is the single-number value of $\left\{ \{ 2 \mid 6 \} \middle| \{ 6 \mid 10 \} \right\}$?

⋆ Answer:

$$\begin{aligned} G &= \left\{ \{ 2 \mid 6 \} \middle| \{ 6 \mid 10 \} \right\} \\ &= \{ 3 \mid 7 \} \\ &= 4 \end{aligned}$$

Answer of exercise 3.2.17

Answer Not Provided

Answer of exercise 3.2.18

Question:

What is the simplified value of $\left\{ \{ 0 \mid 1 \} \middle| \{ 1 \mid 2 \} \right\}$?

⋆ Answer:

$$\begin{aligned} G &= \left\{ \{ 0 \mid 1 \} \middle| \{ 1 \mid 2 \} \right\} \\ &= \left\{ \frac{1}{2} \middle| 1 + \frac{1}{2} \right\} \\ &= 1 \end{aligned}$$

Answer of exercise 3.2.19

Answer Not Provided

Answer of exercise 3.2.20

Question:

What is the simplified value of $\left\{ \{ -10 \mid -1 \} \middle| \{ 0 \mid 4 \} \right\}$?

⋆ Answer:

$$\begin{aligned} G &= \left\{ \{ -10 \mid -1 \} \,\middle|\, \{ 0 \mid 4 \} \right\} \\ &= \{ -2 \mid 1 \} \\ &= 0 \end{aligned}$$

Answer of exercise 3.2.21

Answer Not Provided

Answer of exercise 3.2.22

Question:

What is the simplified value of $\left\{ \left\{ \{ 3 \mid 5 \} \,\middle|\, \{ 11 \mid 13 \} \right\} \,\middle|\, \left\{ \{ 20 \mid 33 \} \,\middle|\, \{ 42 \mid 2176 \} \right\} \right\}$

(The eight numbers used here were supplied by one of the authors' children.)

⋆ Answer:

$$\begin{aligned} G &= \left\{ \left\{ \{ 3 \mid 5 \} \,\middle|\, \{ 11 \mid 13 \} \right\} \,\middle|\, \left\{ \{ 20 \mid 33 \} \,\middle|\, \{ 42 \mid 2176 \} \right\} \right\} \\ &= \left\{ \{ 4 \mid 12 \} \,\middle|\, \{ 21 \mid 43 \} \right\} \\ &= \{ 5 \mid 22 \} \\ &= 6 \end{aligned}$$

Answer of exercise 3.2.23

Answer Not Provided

Answer of exercise 3.2.24

Question:

Use a direct proof to show that $\{ 0 \mid 1, 1+* \}$ is equal to $\frac{1}{2}$.

⋆ Answer:

$$\left\{ 0 \,\middle|\, 1, 1+* \right\} - \frac{1}{2} = \left\{ 0 \,\middle|\, 1, 1+* \right\} + \left\{ -1 \,\middle|\, 0 \right\}$$
$$= \left\{ 0 - \frac{1}{2}, \left\{ 0 \,\middle|\, 1, 1+* \right\} - 1 \,\middle|\, 1 - \frac{1}{2}, 1+* - \frac{1}{2}, \left\{ 0 \,\middle|\, 1, 1+* \right\} + 0 \right\}$$
$$= \left\{ -\frac{1}{2}, \left\{ 0 \,\middle|\, 1, 1+* \right\} - 1 \,\middle|\, \frac{1}{2}, \frac{1}{2} + *, \left\{ 0 \,\middle|\, 1, 1+* \right\} \right\}$$

Now we check that all options are losing:

- $-\frac{1}{2} < 0$, so it's in $\mathcal{R}$.
- In $\left\{ 0 \,\middle|\, 1, 1+* \right\} - 1$, R can move to $1 - 1 = 0$.
- $\frac{1}{2} > 0$, so it's in $\mathcal{L}$.
- $\frac{1}{2} > *$, so $\frac{1}{2} + * > 0$, and it's in $\mathcal{L}$.
- In $\left\{ 0 \,\middle|\, 1, 1+* \right\}$, L can move to 0.

Thus, $\left\{ 0 \,\middle|\, 1, 1+* \right\} - \frac{1}{2}$, so $\left\{ 0 \,\middle|\, 1, 1+* \right\} = \frac{1}{2}$.

Answer of exercise 3.2.25

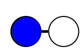
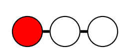
Answer Not Provided

Answer of exercise 3.2.26

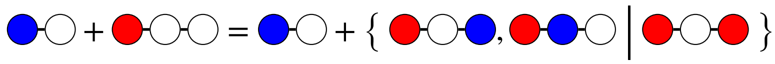
Answer Not Provided

Answer of exercise 3.2.27

Question:

Find the number value of the sum of COL positions  +  .

⋆ Answer:



$$
\begin{aligned}
&= -1 + \left\{\, 0, -1 \,\middle|\, 1 \,\right\} \\
&= -1 + \left\{\, 0 \,\middle|\, 1 \,\right\} \text{ (by simplest numbers)} \\
&= -1 + \frac{1}{2} \\
&= -\frac{1}{2}
\end{aligned}
$$

Answer of exercise 3.2.28

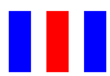
Answer Not Provided

Answer of exercise 3.2.29

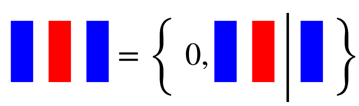
Answer Not Provided

Answer of exercise 3.2.30

Question:

What is the value of  ? You may need to use the result from exercise 3.2.23 .

⋆ Answer:



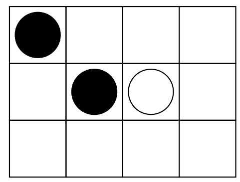
$$
\begin{aligned}
&= \left\{\, 0, \left\{\, 0, -1 \,\middle|\, 1, 0 \,\right\} \,\middle|\, 1 \,\right\} \\
&= \left\{\, 0, \left\{\, 0 \,\middle|\, 0 \,\right\} \,\middle|\, 1 \,\right\} \\
&= \left\{\, 0, * \,\middle|\, 1 \,\right\} \\
&= \frac{1}{2}
\end{aligned}
$$

Answer of exercise 3.2.31

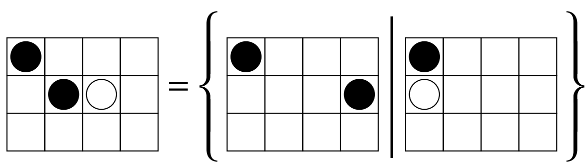
Answer Not Provided

Answer of exercise 3.2.32

Question:

What is the value of this KONANE position:  ?

⋆ Answer:



$$= \{\, 0 \mid 1 \,\}$$
$$= \frac{1}{2}$$

Answer of exercise 3.2.33

Answer Not Provided

Answer of exercise 3.2.34

Answer Not Provided

Answer of exercise 3.3.0

Question:

Rewrite $\{\, 2 \mid 0 \,\}$ in the form $a \pm x$.

⋆ Answer:

Using the formula in the chapter,

$$\left\{ 2 \mid 0 \right\} = a \pm x$$
$$= \frac{b+c}{2} \pm \left(\frac{b-c}{2}\right)$$
$$= \frac{2}{2} \pm \frac{2-0}{2}$$
$$= 1 \pm 1$$

Answer of exercise 3.3.1

Answer Not Provided

Answer of exercise 3.3.2

Question:

Rewrite $\left\{ -100 \mid -300 \right\}$ in the form $a \pm x$.

★ Answer:

Using the formula in the chapter,

$$\left\{ -100 \mid -300 \right\} = a \pm x$$
$$= \frac{b+c}{2} \pm \left(\frac{b-c}{2}\right)$$
$$= \frac{-400}{2} \pm \frac{200}{2}$$
$$= -200 \pm 100$$

Answer of exercise 3.3.3

Answer Not Provided

Answer of exercise 3.3.4

Question:

Rewrite the sum $\left\{ 0 \mid 7 \right\} + \left\{ 7 \mid 0 \right\}$ as a single position of the form $\left\{ x \mid y \right\}$.

⋆ Answer:

$$\begin{aligned}\{\,0\mid 7\,\}+\{\,7\mid 0\,\} &= 1+3.5\pm 3.5\\ &= 4.5\pm 3.5\\ &= \{\,1\mid 8\,\}\end{aligned}$$

Answer of exercise 3.3.5

Answer Not Provided

Answer of exercise 3.3.6

Question:

Rewrite the sum $\{\,2\mid 5\,\}+\{\,5\mid 2\,\}$ as a single position of the form $\{\,x\mid y\,\}$.

⋆ Answer:

$$\begin{aligned}\{\,2\mid 5\,\}+\{\,5\mid 2\,\} &= 3+3\frac{1}{2}\pm\frac{3}{2}\\ &= 6\frac{1}{2}\pm\frac{3}{2}\\ &= \{\,8\mid 5\,\}\end{aligned}$$

Answer of exercise 3.3.7

Answer Not Provided

Answer of exercise 3.3.8

Question:

Rewrite the sum $\{\,-2\mid -100\,\}+\{\,-100\mid -2\,\}$ as a single position of the form $\{\,x\mid y\,\}$.

⋆ Answer:

$$\begin{aligned}\{\, -2 \mid -100 \,\} + \{\, -100 \mid -2 \,\} &= -51 \pm 49 - 3 \\ &= -54 \pm 49 \\ &= \{\, -5 \mid -103 \,\}\end{aligned}$$

Answer of exercise 3.3.9

Answer Not Provided

Answer of exercise 3.3.10

Question:

Rewrite the sum $G = \{\, 6 \mid 2 \,\} + \{\, -2 \mid -6 \,\}$ as a sum of a number and switches and then find $o(G)$.

⋆ Answer:

$$\begin{aligned}G &= \{\, 6 \mid 2 \,\} + \{\, -2 \mid -6 \,\} \\ &= 4 \pm 2 + -4 \pm 2 \\ &= 0 \pm 2 \pm 2 \\ &= 0\end{aligned}$$

$0 \in \mathcal{P}$, so $o(G) = \mathcal{P}$.

Answer of exercise 3.3.11

Answer Not Provided

Answer of exercise 3.3.12

Question:

Rewrite the sum $G = \{\, -1 \mid -2 \,\} + \{\, 4 \mid 3 \,\}$ as a sum of a number and switches and then find $o(G)$.

⋆ Answer:

$$\begin{aligned} G &= \{ -1 \mid -2 \} + \{ 4 \mid 3 \} \\ &= -1.5 \pm \frac{1}{2} + 3.5 \pm \frac{1}{2} \\ &= 2 \pm \frac{1}{2} \pm \frac{1}{2} \\ &= 2 \\ &\in \mathcal{L} \end{aligned}$$

Answer of exercise 3.3.13

Answer Not Provided

Answer of exercise 3.3.14

Question:

Rewrite $G = -2 \pm 3 \pm 1$ without the $\pm$ symbol. (Hint: your position will have the form $\left\{ \{ a \mid b \} \,\middle|\, \{ c \mid d \} \right\}$.)

⋆ Answer:

$$\begin{aligned} -2 \pm 3 \pm 1 &= \{ 1 \pm 1 \mid -5 \pm 1 \} \\ &= \left\{ \{ 2 \mid 0 \} \,\middle|\, \{ -4 \mid -6 \} \right\} \end{aligned}$$

Answer of exercise 3.3.15

Answer Not Provided

Answer of exercise 3.3.16

Question:

Rewrite $G = 2\frac{1}{2} \pm 4 \pm 3$ without the $\pm$ symbol.

⋆ Answer:

$$2\frac{1}{2} \pm 4 \pm 3 = \left\{\ 6\frac{1}{2} \pm 3 \ \middle|\ -1\frac{1}{2} \pm 3\ \right\}$$
$$= \left\{\ \left\{\ 9\frac{1}{2} \ \middle|\ 3\frac{1}{2}\ \right\} \ \middle|\ \left\{\ 1\frac{1}{2} \ \middle|\ -4\frac{1}{2}\ \right\}\ \right\}$$

Answer of exercise 3.3.17

Answer Not Provided

Answer of exercise 3.3.18

Question:

Rewrite $G = \left\{\ \{\ 12 \mid 6\ \} \ \middle|\ \{\ 4 \mid -2\ \}\ \right\}$ as a sum of two switches.

⋆ Answer:

$$\left\{\ \{\ 12 \mid 6\ \} \ \middle|\ \{\ 4 \mid -2\ \}\ \right\} = \{\ 9 \pm 3 \mid 1 \pm 3\ \}$$
$$= 5 \pm 4 \pm 3$$

Answer of exercise 3.3.19

Answer Not Provided

Answer of exercise 3.3.20

Question:

Rewrite $G = \left\{\ \{\ 4 \mid 3\ \} \ \middle|\ \{\ 2 \mid 1\ \}\ \right\}$ as a sum of two switches.

⋆ Answer:

$$\left\{\ \{\ 4 \mid 3\ \} \ \middle|\ \{\ 2 \mid 1\ \}\ \right\} = \left\{\ 3\frac{1}{2} \pm \frac{1}{2} \ \middle|\ 1\frac{1}{2} \pm \frac{1}{2}\ \right\}$$
$$= 2 \pm 1\frac{1}{2} \pm \frac{1}{2}$$

Answer of exercise 3.3.21

Answer Not Provided

Answer of exercise 3.3.22

Question:

Simplify $G = \left\{ \{ 2 \mid 4 \} \,\middle|\, \{ -2 \mid 0 \} \right\}$ to a switch or sum of switches.

⋆ Answer:

$$\begin{aligned} \left\{ \{ 2 \mid 4 \} \,\middle|\, \{ -2 \mid 0 \} \right\} &= \{ 3 \mid -1 \} \\ &= 1 \pm 2 \end{aligned}$$

Answer of exercise 3.3.23

Answer Not Provided

Answer of exercise 3.3.24

Question:

Simplify $G = \left\{ \{ 14 \mid 50 \} \,\middle|\, \{ 4 \mid 100 \} \right\}$ to a switch or sum of switches.

⋆ Answer:

$$\begin{aligned} \left\{ \{ 14 \mid 50 \} \,\middle|\, \{ 4 \mid 100 \} \right\} &= \{ 15 \mid 5 \} \\ &= 10 \pm 5 \end{aligned}$$

Answer of exercise 3.3.25

Answer Not Provided

Answer of exercise 3.3.26

Answer Not Provided

Answer of exercise 3.3.27

Question:

$G = \left\{ \{ 7 \mid 5 \} \,\middle|\, \{ -3 \mid x \} \right\}$. Is it possible for G to be the sum of two switches? If so, find x. If not, explain why not.

⋆ Answer:

In order for G to be the sum of two switches, the left and right options have to be switches with the same heat. The left option, $\{ 7 \mid 5 \} = 6 \pm 1$, so the right must also have the form $a \pm 1$. $-3 - 1 = -4$, so that side must be -4 ± 1. Then $G = \{ 6 \pm 1 \mid -4 \pm 1 \} = 1 \pm 5 \pm 1$, which is a legitimate sum of switches. Thus, $\{ -3 \mid x \} = -4 \pm 1$, meaning $x = -5$.

Answer of exercise 3.3.28

Answer Not Provided

Answer of exercise 3.3.29

Question:

$G = \left\{ \{ 1 \mid 0 \} \,\middle|\, \{ x \mid -2 \} \right\}$. Is it possible for G to be the sum of two switches? If so, find x. If not, explain why not.

⋆ Answer:

In order for G to be the sum of two switches, the left and right options have to be switches with the same heat. The left option, $\{ 1 \mid 0 \} = \frac{1}{2} \pm \frac{1}{2}$, so the right must also have the form $a \pm \frac{1}{2}$. $-2 + \frac{1}{2} = -1\frac{1}{2}$, so that side must be $-1\frac{1}{2} \pm \frac{1}{2}$. Then $G = \left\{ \frac{1}{2} \pm \frac{1}{2} \,\middle|\, -1\frac{1}{2} \pm \frac{1}{2} \right\} = -\frac{1}{2} \pm 1 \pm \frac{1}{2}$, which is a legitimate sum of switches. Thus, $\{ x \mid -2 \} = -1\frac{1}{2} \pm \frac{1}{2}$, meaning $x = -1$.

Answer of exercise 3.3.30

Answer Not Provided

Answer of exercise 3.3.31

Question:

Simplify $G = \left\{ \{ 11 \mid 15 \} \,\middle|\, \{ 7 \mid 15 \} \right\} + \left\{ \{ \mid -10 \} \,\middle|\, \{ -20 \mid -16 \} \right\}$ by expressing it as a sum of switches.

⋆ Answer:

$$\begin{aligned} G &= \left\{ \{ 11 \mid 15 \} \,\middle|\, \{ 7 \mid 15 \} \right\} + \left\{ \{ \mid -10 \} \,\middle|\, \{ -20 \mid -16 \} \right\} \\ &= \{ 12 \mid 8 \} + \{ -11 \mid -17 \} \\ &= 10 \pm 2 + -14 \pm 3 \\ &= -4 \pm 3 \pm 2 \end{aligned}$$

Answer of exercise 3.3.32

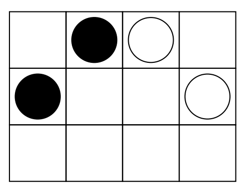
Answer Not Provided

Answer of exercise 3.3.33

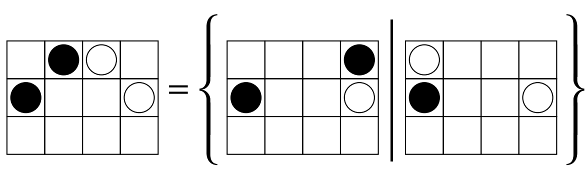
Answer Not Provided

Answer of exercise 3.3.34

Question:

What is the value of this KONANE position:  ?

⋆ Answer:



$$\begin{aligned} &= \{ 1 \mid -1 \} \\ &= \pm 1 \end{aligned}$$

Answer of exercise 3.3.35

Answer Not Provided

Answer of exercise 3.3.36

Answer Not Provided

Answer of exercise 3.3.37

Answer Not Provided

Answer of exercise 3.3.38

Answer Not Provided

Answer of exercise 3.3.39

Question:

Use a direct proof to show that $- \pm x = \pm x$ for any non-negative number x.

⋆ Answer:

$$\begin{aligned} - \pm x &= -\{\, x \mid -x \,\} \\ &= \{\, --x \mid -x \,\} \\ &= \{\, x \mid -x \,\} \\ &= \pm x \end{aligned}$$

Answer of exercise 3.3.40

Answer Not Provided

Answer of exercise 3.3.41

Question:

Consider $G = \left\{ \{\, w \mid x \,\} \,\middle|\, \{\, y \mid z \,\} \right\}$ where w, x, y, and z are all numbers. Using your answer from exercise 3.3.17 , fill in the boxes below with comparison operators to give four necessary and sufficient conditions for G to be the sum of two switches:

- $w \square x \square y \square z$ and
- $w - x \square y - z$

⋆ Answer:

- $w \geq x \geq y \geq z$ and
- $w - x = y - z$

Answer of exercise 3.3.42

Answer Not Provided

Answer of exercise 3.5.0

Answer Not Provided

Answer of exercise 3.5.1

Answer Not Provided

Answer of exercise 3.5.2

Question:

Since $\uparrow > 0$ we know that $G + \uparrow > G$ for any game G. So, in particular, $\uparrow + \uparrow > \uparrow$. Prove that $\uparrow + \uparrow$, denoted $\Uparrow$ is an infinitesimal.

⋆ Answer:

Proof. It suffices to show that $1/2^n - \Uparrow > 0$ for any $n \in \mathbb{N}^+$.

$$\begin{aligned} 1/2^n - \Uparrow &= 1/2^n + \Downarrow \\ &= 1/2^n + \{ * \mid 0 \} + \{ * \mid 0 \} \end{aligned}$$

Note that if R moves one of the $\downarrow$'s to 0, then that would leave $1/2^n - \uparrow > 0$. And if they play on $1/2^n$ then the game will move to

$$1/2^{n-1} + \{ * \mid 0 \} + \{ * \mid 0 \},$$

to which L can respond with $1/2^{n-1} - \uparrow$, which we've seen is in $\mathcal{L}$. Therefore $1/2^n - \Uparrow > 0$. □

Answer of exercise 3.5.3

Answer Not Provided

Answer of exercise 3.5.4

Question:

Show that the game $+_1 + *k$ is bigger than $*k$.

★ Answer:

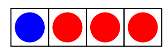
$+_1 + *k - *k = +_1 > 0$

Answer of exercise 3.5.5

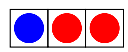
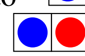
Answer Not Provided

Answer of exercise 3.5.6

Question:

What is the value of the CLOBBER position  ?

★ Answer:

R can move to 0 and L can move to , from which R can move to 0 and L can move to , which has value $*$. So the original position has value $\left\{ \{ * \mid 0 \} \,\middle|\, 0 \right\} = \{ \downarrow \mid 0 \}$.

Answer of exercise 3.5.7

Answer Not Provided

Answer of exercise 3.5.8

Answer Not Provided

Answer of exercise 3.5.9

Answer Not Provided

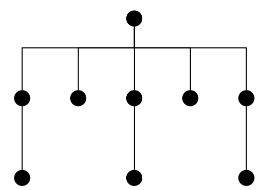
Answer of exercise 3.5.10

Answer Not Provided

Answer of exercise 4.1.0

Question:

Is the following impartial game tree a They-Love-Me-They-Love-Me-Not (TLMTLMN) position? Find the outcome class and the value of the tree.



⋆ Answer:

This is not a TLMNTLMN position because not all possible game paths have the same length parity. The leaf on the far left is two moves from the initial position; the leaf to the right of that is a direct option from the start. The initial position is in $\mathcal{N}$ and has a value of $*2$.

Answer of exercise 4.1.1

Answer Not Provided

Answer of exercise 4.1.2

Answer Not Provided

Answer of exercise 4.1.3

Answer Not Provided

Answer of exercise 4.1.4

Question:

What is the value of a BRUSSELS SPROUTS starting position with one node?

⋆ Answer:

$k = 1$, so the number of moves will be $5k - 2 = 5 - 2 = 3$. This is odd, so the first player will win, meaning the game equals $*$.

Answer of exercise 4.1.5

Answer Not Provided

Answer of exercise 4.1.6

Answer Not Provided

Answer of exercise 4.1.7

Answer Not Provided

Answer of exercise 4.1.8

Answer Not Provided

Answer of exercise 4.2.0

Question:

What is the outcome class of $G = 5 + \Uparrow + *6$?

⋆ Answer:

$G \in \mathcal{L}$.

Answer of exercise 4.2.1

Answer Not Provided

Answer of exercise 4.2.2

Answer Not Provided

Answer of exercise 4.2.3

Question:

What is the outcome class of $G = -10 \pm 5$?

⋆ Answer:

$t = 5$. Since this is less than $- -10$ ($t < -x$), $G \in \mathcal{R}$.

Answer of exercise 4.2.4

Answer Not Provided

Answer of exercise 4.2.5

Question:

What is the outcome class of $G = 3 \pm 3$?

⋆ Answer:

$t = 3$. Since this is equal to 3 ($t = x$) and n is odd, $G \in \mathcal{N}$.

Answer of exercise 4.2.6

Answer Not Provided

Answer of exercise 4.2.7

Question:

What is the outcome class of $G = \pm 6 \pm 6$?

⋆ Answer:

$t = 6 - 6 = 0$. Since both x and t are zero, $G \in \mathcal{P}$.

Answer of exercise 4.2.8

Answer Not Provided

Answer of exercise 4.2.9

Question:

What is the outcome class of $G = -2 \pm 7 \pm 3$?

⋆ Answer:

$t = 7 - 3 = 4$. Since this is greater than 2 ($t > |x|$), $G \in \mathcal{N}$.

Answer of exercise 4.2.10

Answer Not Provided

Answer of exercise 4.2.11

Question:

What is the outcome class of $G = \pm 6 \pm 3 \pm 3$?

⋆ Answer:

$t = 6 - 3 + 3 = 6$. Here $x = 0$, so $t > x$ and $G \in \mathcal{N}$.

Answer of exercise 4.2.12

Answer Not Provided

Answer of exercise 4.2.13

Question:

What is the outcome class of $G = 7 \pm 6 \pm 5 \pm 4 \pm 3$?

⋆ Answer:

$t = 6 - 5 + 4 - 3 = 2$, $x = 7$, and n is even. Here $x > 0$, and $t < x$ so $G \in \mathcal{L}$.

Answer of exercise 4.2.14

Answer Not Provided

Answer of exercise 4.2.15

Answer Not Provided

Answer of exercise 4.2.16

Answer Not Provided

Answer of exercise 4.2.17

Question:

What is the outcome class of $G = 1 \pm 1 + \uparrow + *$?

⋆ Answer:

If L goes first, they should play on the switch, resulting in a value of $2 + \uparrow + *$, which is clearly in $\mathcal{L}$. If R goes first, after their turn, the game will have a value of $\uparrow + * \in \mathcal{N}$. Since L goes next, they will be able to win. Thus, L has a winning strategy going both first and second, so $G \in \mathcal{L}$.

Answer of exercise 4.2.18

Question:

What is the outcome class of $G = \pm 4 \pm 4 + \downarrow$?

⋆ Answer:

$G = \downarrow \in \mathcal{R}$.

Answer of exercise 4.2.19

Answer Not Provided

Answer of exercise 4.2.20

Answer Not Provided

Answer of exercise 4.2.21

Question:

What is the outcome class of $G = 4 \pm 4 + \downarrow + *$?

$\star$ Answer:

If L goes first, they will play on the switch, resulting in a value of $8 + \downarrow + *$, which is clearly in $\mathcal{L}$. If R goes first, they will take the switch, so that it's L's turn on $\downarrow + *$, which is in $\mathcal{N}$. Since L goes next, they will be able to win. Thus, L has a winning strategy going both first and second, so $G \in \mathcal{L}$.

Answer of exercise 4.2.22

Answer Not Provided

Answer of exercise 4.2.23

Answer Not Provided

Answer of exercise 4.2.24

Question:

What is the outcome class of $G = 2 \pm 7 \pm 1 + \Uparrow + *$?

$\star$ Answer:

If L goes first, after the switches, the game will have a value of $8 + \Uparrow + *$, which is clearly in $\mathcal{L}$. If R goes first, L will complete the switches at a value of $-4 + \Uparrow + * \in \mathcal{R}$. Both players win going first, so $G \in \mathcal{N}$.

Answer of exercise 4.2.25

Answer Not Provided

Answer of exercise 4.2.26

Question:

What is the outcome class of $G = \pm 4 \pm 3 + \downarrow + *8$?

⋆ Answer:

With L going first, after two moves, the value will be $1 + \downarrow + *8 \in \mathcal{L}$. If R goes first, then after two moves the value will be $-1 + \downarrow + *8 \in \mathcal{R}$. Thus, $G \in \mathcal{N}$.

Answer of exercise 4.2.27

Answer Not Provided

Answer of exercise 4.2.28

Answer Not Provided

Answer of exercise 4.2.29

Question:

What is the outcome class of $G = \pm \uparrow$?

⋆ Answer:

$G = \pm \uparrow = \left\{ \uparrow \middle| \downarrow \right\}$. The first player will always win here, so $G \in \mathcal{N}$. (Note: $\pm \uparrow = *$.)

Answer of exercise 4.2.30

Answer Not Provided

Answer of exercise 4.3.0

Answer Not Provided

Answer of exercise 4.3.1

Question:

Consider $G = \left\{ \left\{ 4 \mid 2 \right\} \middle| -2 \right\}$. Find the left and right stops and the confusion intervals of G, $2 \times G$, $3 \times G$, and $4 \times G$. Is this information enough to find the mean value of G, $m(G)$? If so, find $m(G)$. Additionally, find the temperature, $t(G)$. (You do not need to show calculations of $\breve{G}_t$ that reach the incorrect temperature.)

⋆ Answer:

In multiple copies of G, it's better for R to play in G than in $\{ 4 \mid 2 \}$. Thus:

- $LS(G) = 2_R$, $RS(G) = -2_R$, so the confusion interval of G is: $[-2, 2)$.
- $LS(2 \times G) = 2_L$, $RS(2 \times G) = 0_R$, so the confusion interval of $2 \times G$ is: $[0, 2]$.
- $LS(3 \times G) = 4_L$, $RS(3 \times G) = 0_L$, so the confusion interval of $3 \times G$ is: $(0, 4]$.
- $LS(4 \times G) = 2_R$, $RS(4 \times G) = 2_L$, so the confusion interval of $4 \times G$ is: $(2, 2)$.

Since the interval converges, $m(G) = {}^{LS(4\times G)}/_4 = {}^1/_2$.
The correct temperature is $t = 5/2$:

$$\begin{aligned}
\breve{G}_{5/2} &= \left\{ \left\{ 4 \,\breve{|}\, 2 \right\}_{5/2} - 5/2 \,\middle|\, -2 + 5/2 \right\} \\
&= \left\{ \left\{ 4 - 5/2 \mid 2 + 5/2 \right\} - 5/2 \,\middle|\, 1/2 \right\} \\
&= \left\{ \left\{ 3/2 \mid 9/2 \right\} - 5/2 \,\middle|\, 1/2 \right\} \\
&= \left\{ 2 - 5/2 \mid 1/2 \right\} \\
&= \left\{ -1/2 \mid 1/2 \right\} \\
&= 0
\end{aligned}$$

$\breve{G}_{5/2}$ is a number, so the stops are equal. They're also inside the confusion interval of $[-2, 2)$, so the temperature is $5/2$.

Answer of exercise 4.3.2

Answer Not Provided

Answer of exercise 4.3.3

Question:

Consider $G = \left\{ 100 \,\middle|\, \{ -10 \mid -20 \} \right\}$. Find the left and right stops and the confusion intervals of G, $2 \times G$, $3 \times G$, and $4 \times G$. Is this information enough to find the mean value of G, $m(G)$? If so, find $m(G)$. Additionally, find the temperature, $t(G)$. (You do not need to show calculations of $\breve{G}_t$ that reach the incorrect temperature.)

⋆ Answer:

In multiple copies of G, it's better for L to play in G than in $\{ -10 \mid -20 \}$. Thus:

- $LS(G) = 100_L$, $RS(G) = -10_L$, so the confusion interval of G is: $(-10, 100]$.
- $LS(2 \times G) = 90_L$, $RS(2 \times G) = 80_R$, so the confusion interval of $2 \times G$ is: $[80, 90]$.
- $LS(3 \times G) = 180_R$, $RS(3 \times G) = 70_R$, so the confusion interval of $3 \times G$ is: $[70, 180)$.
- $LS(4 \times G) = 170_R$, $RS(4 \times G) = 170_L$, so the confusion interval of $4 \times G$ is: $(170, 170)$.

Since the interval converges, $m(G) = {}^{LS(4 \times G)}/_4 = {}^{170}/_4 = 42 + {}^1/_2$.
It may take a couple tries, but the correct temperature is 50:

$$\begin{aligned}
\breve{G}_{50} &= \left\{ 100 - 50 \,\middle|\, \{ -10 \,\breve{|}\, -20 \}_{50} + 50 \right\} \\
&= \left\{ 50 \,\middle|\, \{ -10 - 50 \mid -20 + 50 \} + 50 \right\} \\
&= \left\{ 50 \,\middle|\, \{ -60 \mid 30 \} + 50 \right\} \\
&= \{ 50 \mid 0 + 50 \} \\
&= 50 + *
\end{aligned}$$

The stops are both 50, which is within the confusion interval of $(-10, 100]$. $t(G) = 50$.

Note: In a prior version of this text, the temperature was incorrectly calculated to be $115/2$.

Answer of exercise 4.3.4

Answer Not Provided

Answer of exercise 4.3.5

Question:

Consider $G = \left\{ 2 \,\middle|\, \{ 0 \mid -2 \} \right\}$. Find the left and right stops and the confusion intervals of $G, 2 \times G$, $3 \times G$, and $4 \times G$. Is this information enough to find the mean value of G, $m(G)$? If so, find $m(G)$. Additionally, find the temperature, $t(G)$. (You do not need to show calculations of $\breve{G}_t$ that reach the incorrect temperature.)

⋆ Answer:

In multiple copies of G, it's better for L to play in G than in $\{ 0 \mid -2 \}$. Thus:

- $LS(G) = 2_L$, $RS(G) = 0_L$, so the confusion interval of G is: $(0, 2]$.
- $LS(2 \times G) = 2_L$, $RS(2 \times G) = 0_R$, so the confusion interval of $2 \times G$ is: $[0, 2]$.
- $LS(3 \times G) = 2_R$, $RS(3 \times G) = 0_R$, so the confusion interval of $3 \times G$ is: $[0, 2)$.
- $LS(4 \times G) = 2_R$, $RS(4 \times G) = 2_L$, so the confusion interval of $4 \times G$ is: $(2, 2)$.

Since the interval converges, $m(G) = {}^{LS(4\times G)}\!/_4 = {}^2\!/_4 = {}^1\!/_2$.

It may take a couple tries, but the correct temperature is $3/2$:

$$\begin{aligned}\breve{G}_{3/2} &= \left\{ 2-\frac{3}{2} \,\middle|\, \{\,0\,|\breve{\ }{-2}\,\}_{3/2}+\frac{3}{2} \right\} \\ &= \left\{ 1/2 \,\middle|\, \left\{ 0-\frac{3}{2} \,\middle|\, -2+\frac{3}{2} \right\}+3/2 \right\} \\ &= \left\{ 1/2 \,\middle|\, \{\,-3/2 \,|\, -1/2\,\}+3/2 \right\} \\ &= \{\,1/2 \,|\, -1+3/2\,\} \\ &= \{\,1/2 \,|\, 1/2\,\} \\ &= 1/2+*\end{aligned}$$

The stops are both $1/2$, which is within the confusion interval of $(0, 2]$. $t(G) = 3/2$.

Answer of exercise 4.3.6

Answer Not Provided

Answer of exercise 4.3.7

Question:

Consider $G = \left\{ \{\,7 \,|\, 0\,\} \,\middle|\, -5 \right\}$. Find the left and right stops and the confusion intervals of G, $2 \times G$, $3 \times G$, and $4 \times G$. Determine $m(G)$ from the limit of the stops of $n \times G$, then find $t(G)$.

⋆ Answer:

In multiple copies of G, it's better for R to play in $\{\,7 \,|\, 0\,\}$ than in another copy of G. Thus:

- $LS(G) = 0_R$, $RS(G) = -5_R$, so the confusion interval of G is: $[-5, 0)$.
- $LS(2 \times G) = 0_R$, $RS(2 \times G) = -5_R$, so the confusion interval of $2 \times G$ is: $[-5, 0)$.
- $LS(3 \times G) = 0_R$, $RS(3 \times G) = -5_R$, so the confusion interval of $3 \times G$ is: $[-5, 0)$.

- $LS(4 \times G) = 0_R$, $RS(4 \times G) = -5_R$, so the confusion interval of $4 \times G$ is: $[-5, 0)$.

The interval does not seem to converge after a finite number of iterations. In general, because R will always play in $\{ 7 \mid 0 \}$ instead of another copy of G, $LS(n \times G)$ will always equal $(0n)_R = 0_R$ and $RS(n \times G)$ will always be $(-5 + 0(n-1))_R = -5_R$. The limits as $n \to \infty$ of both must equal, so we can use either:

$$\begin{aligned} m(G) &= \lim_{n\to\infty} \frac{LS(n \times G)}{n} \\ &= \lim_{n\to\infty} 0 \\ &= 0, \text{ and} \\ t(G) &= \max(LS(G) - m(G), m(G) - RS(G)) \\ &= \max\left(0 - 0, 0 - -5\right) \\ &= \max\left(0, 5\right) \\ &= 5 \end{aligned}$$

Answer of exercise 4.3.8

Answer Not Provided

Answer of exercise 4.3.9

Question:

Consider $G = \left\{ \{ -10 \mid -20 \} \,\middle|\, -25 \right\}$. Find the left and right stops and the confusion intervals of G, $2 \times G$, $3 \times G$, and $4 \times G$. Determine $m(G)$ from the limit of the stops of $n \times G$, then find $t(G)$.

★ Answer:

In multiple copies of G, it's better for R to play in $\{ -10 \mid -20 \}$ than in another copy of G. Thus:

- $LS(G) = -20_R$, $RS(G) = -25_R$, so the confusion interval of G is: $[-25, -20)$.
- $LS(2 \times G) = -40_R$, $RS(2 \times G) = -45_R$, so the confusion interval of $2 \times G$ is: $[-45, -40)$.

- $LS(3\times G) = -60_R$, $RS(3\times G) = -65_R$, so the confusion interval of $3\times G$ is: $[-65, -60)$.
- $LS(4\times G) = -80_R$, $RS(4\times G) = -85_R$, so the confusion interval of $4\times G$ is: $[-85, -80)$.

The interval does not seem to converge after a finite number of iterations. In general, because R will always play in $\{\, -10 \mid -20 \,\}$ instead of another copy of G, $LS(n \times G)$ will always equal $(-20n)_R$ and $RS(n \times G)$ will always be $(-25 - 20(n-1))_R = (-20n-5)_R$. The limits as $n \to \infty$ of both must equal, so we can use either:

$$\begin{aligned}
m(G) &= \lim_{n\to\infty} \frac{LS(n\times G)}{n} \\
&= \lim_{n\to\infty} \frac{-20n}{n} \\
&= \lim_{n\to\infty} -20 \\
&= -20, \text{ and} \\
t(G) &= \max(LS(G) - m(G), m(G) - RS(G)) \\
&= \max\left(-20 - -20, -20 - -25\right) \\
&= \max\left(0, 5\right) \\
&= 5
\end{aligned}$$

Answer of exercise 4.3.10

Answer Not Provided

Answer of exercise 4.3.11

Question:

Consider $G = \left\{ \{\, 10 \mid 9 \,\} \;\middle|\; \{\, 2 \mid 0 \,\} \right\}$. Use the tactics we learned to find $m(G)$ and the temperature of $t(G)$. Make sure you explain the optimal strategies each player uses while describing the stops!

⋆ Answer:

In multiple copies of G, it's better for both players to play in G than in the individual components. Thus:

- $LS(G) = 9_R$, $RS(G) = 2_L$, so the confusion interval of G is: $(2, 9)$.
- $LS(2 \times G) = 11_R$, $RS(2 \times G) = 11_L$, so the confusion interval of $2 \times G$ is: $(11, 11)$.

We've already converged at $n = 2$, so $m(G) = {}^{LS(2\times G)}\!/_2 = {}^{11}\!/_2 = 5 + \frac{1}{2}$. Then,

$$\begin{aligned} t(G) &= \max(LS(G) - m(G), m(G) - RS(G)) \\ &= \max\left(9 - \left(5 + \frac{1}{2}\right), 5 + \frac{1}{2} - 2\right) \\ &= \max\left(3 + \frac{1}{2}, 3 + \frac{1}{2}\right) \\ &= 3 + \frac{1}{2} \end{aligned}$$

Answer of exercise 4.3.12

Answer Not Provided

Answer of exercise 4.3.13

Question:

Consider $G = \left\{ \{ 5 \mid 3 \} \,\middle|\, \{ 2 \mid 1 \} \right\}$. Use the tactics we learned to find $m(G)$ and the temperature of $t(G)$. Make sure you explain the optimal strategies each player uses while describing the stops!

⋆ Answer:

Let $H = \{ 5 \mid 3 \} = 4 \pm 1$ and $J = \{ 2 \mid 1 \} = 1 + \frac{1}{2} \pm \frac{1}{2}$. In multiple copies of G, H, and J:

- Both players will prefer to play in H before J, as it is a switch with higher temperature.
- L will choose to play on G over H, as $RS(2 \times H) = 8 > 7 = RS(5 + G)$.
- R has no preference between playing on G or H, as $LS(H + J) = LS(G + 3) = 6$. Thus, in our analysis, we will presume that they play on G when given the choice.

Thus, overall, both players will choose to play on G first, then H, then J.

- $LS(G) = 3_R$, $RS(G) = 2_L$, so the confusion interval of G is: $(2, 3)$.
- $LS(2 \times G) = 6_R$, $RS(2 \times G) = 5_L$, so the confusion interval of $2 \times G$ is: $(5, 6)$.
- $LS(3 \times G) = 9_R$, $RS(3 \times G) = 8_L$, so the confusion interval of $3 \times G$ is: $(8, 9)$.

The interval does not seem to converge after a finite number of iterations. In general, because of the play strategies explained above, $LS(n \times G)$ will always equal $3n_R$. The limits as $n \to \infty$ of both must equal, so we can use the left one to find $m(G)$:

$$\begin{aligned}
m(G) &= \lim_{n\to\infty} \frac{LS(n \times G)}{n} \\
&= \lim_{n\to\infty} \frac{3n}{n} \\
&= \lim_{n\to\infty} 3 \\
&= 3, \text{ and} \\
t(G) &= \max(LS(G) - m(G), m(G) - RS(G)) \\
&= \max\left(3 - 3, 3 - 2\right) \\
&= \max\left(0, 1\right) \\
&= 1
\end{aligned}$$

Answer of exercise 4.3.14

Answer Not Provided

Answer of exercise 4.4.0

Answer Not Provided

Answer of exercise 4.4.1

Question:

Determine the winner on an empty HEX board of height 2 and width n.

⋆ Answer:

If n is 1 then it's an immediate win for L. If $n = 2$ a simple case analysis shows that the first player can win. If $n > 2$ then R will win.

Answer of exercise 4.4.2

Answer Not Provided

Answer of exercise 4.4.3

Answer Not Provided

Answer of exercise 4.4.4

Question:

Using the above result, show that TIC-TAC-TOE always ends in a tie when played perfectly by both players.

⋆ Answer:

By the above, an empty board is not in $\mathcal{P}$. A case analysis on the three distinct possible first moves - center, center-side, and corner - shows responses of corner, center, and center, respectively, to prevent a first player win. Thus the game is not in $\mathcal{N}$, either. Thus, it should end in a draw.

Answer of exercise 4.4.5

Answer Not Provided

Answer of exercise 5.1.0

Question:

How many 6 character license plates are possible in which the first three characters are letters and the last three are digits?

⋆ Answer:

$26^3 \cdot 10^3 = 17576000$

Answer of exercise 5.1.1

Answer Not Provided

Answer of exercise 5.1.2

Answer Not Provided

Answer of exercise 5.1.3

Question:

Which is a stronger password? Justify.

1. 8 to 10 characters, including uppercase letters, lowercase letters, digits, and 32 other symbols, or
2. Any three to four 4 letter words, of which there are 3996 in the English language

⋆ Answer:

1. Using $26 + 26 + 10 + 32 = 94$ symbols for $8, 9$, or 10 characters allows for $94^8 + 94^9 + 94^{10} \approx 5.4 \times 10^{19}$ possible passwords.
2. Using three or four out of 3996 possible words yields $3996^3 + 3996^4 \approx 2.6 \times 10^{14}$ passwords.

So the shorter password is stronger.

Answer of exercise 5.1.4

Answer Not Provided

Answer of exercise 5.1.5

Question:

How many distinct flushes are possible?

⋆ Answer:

First we choose a suit, then 7 cards from that suit. $\binom{4}{1}\binom{13}{7} = 6864$

Answer of exercise 5.1.6

Answer Not Provided

Answer of exercise 5.1.7

Question:

How many ways could you get three of one rank and four of another?

⋆ Answer:

Choose one rank for the three, and another for the four. $13\binom{4}{3}12\binom{4}{4} = 624$

Answer of exercise 5.1.8

Answer Not Provided

Answer of exercise 5.1.9

Question:

Use the Inclusion/Exclusion Principle to determine how many positive integers less than or equal to 50 are a multiple of 2, 3, or 5.

⋆ Answer:

Call the sets A, B, C respectively. $|A| = 25, |B| = 16, |C| = 10$, $A \cap B$ is the set of multiple of 6, so $|A \cap B| = 8, |A \cap C| = 5, |B \cap C| = 3$, and $|A \cap B \cap C| = 1$ since the only multiple of all three of the primes is 30. So $|A \cup B \cup C| = 25 + 16 + 10 - 8 - 5 - 3 + 1 = 36$.

Answer of exercise 5.1.11

Answer Not Provided

Answer of exercise 5.1.11

Answer Not Provided

D. Bibliography

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Index

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[1] https://www.ctan.org/pkg/nimsticks

[2] https://tex.stackexchange.com/questions/580860/repeat-items-in-appendix-in-alphabetical-order/580891?noredirect=1#comment1461249_580891

[3] https://gordonlesti.com/debian-increase-latex-main-memory-size/

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