Combinatorial Games Crash Course

Kyle Burke

Plymouth State

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Combinatorial Games

Combinatorial Games:

- Two competing players ("Left", "Right") alternate turns
- No randomness
- No hidden information
- If you can't move, you lose. ("Normal" play)

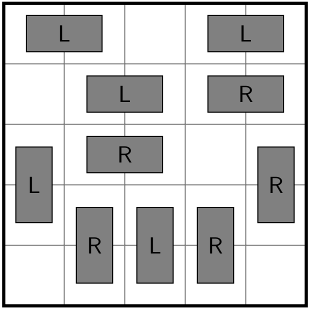
We can describe games based on the moves available to each player.

Cram

Cram:

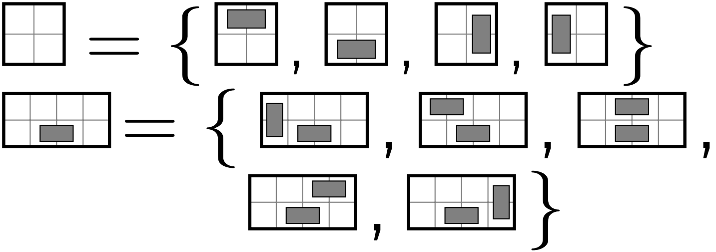
- Start with an empty checkerboard.
- Turn: play a domino on any two empty adjacent squares.

Quick playthrough:

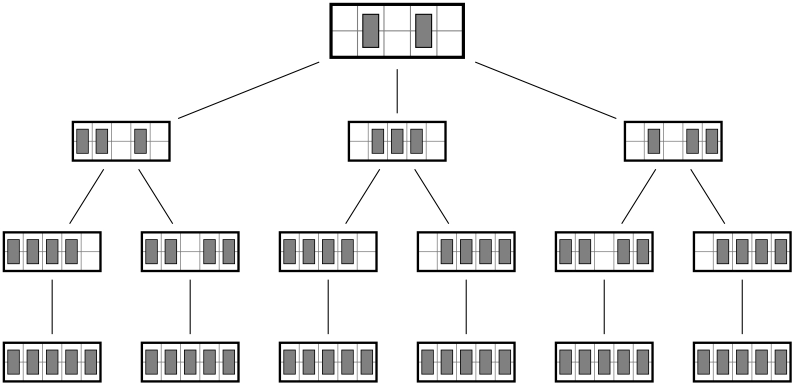


Options

We can abstractly describe each position by the *options*, positions that can be moved to.

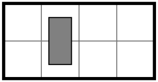
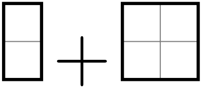


Game Trees



Adding Games

Games break into pieces, which we want to treat independently.

We want:  to act the same as 

Adding Games

So, $G + H$ means on their turn, the player chooses an option from either G or H (not both).

Rigor: $G = \{G_1, \ldots, G_n\}$ and $H = \{H_1, \ldots, H_m\}$, then

$$G + H = \{G_1 + H, \ldots, G_n + H,\ H_1 + G, \ldots, H_m + G\}$$

Most of our definitions are going to revolve around how addition works.

Another game: Nim

Nim:

- Heap(s) of tokens/sticks
- Turn: remove some sticks from any one heap.

Position (as list): [1, 2, 3]
Options of [1, 2, 3]:

- (Take 1 from first) [2, 3],
- (Take 1 from second) [1, 1, 3],
- [1, 3],
- [1, 2, 2],
- [1, 2, 1],
- [1, 2]

[1, 2, 3] = { [2, 3], [1, 1, 3], [1, 3], [1, 2, 2], [1, 2, 1], [1, 2] }

Zero and $*$

$\{\ \}$ seems important. Let's give it a name.

$$\{\ \} := 0$$

Let's do $\{0\}$ too:

$$\{0\} := *$$

Equivalence

What happens when we add 0 to another game, G?

Same as G

Equivalence in general? $G = H$?

$G = H \Leftrightarrow \forall X : G + X$ has the same outcome as $H + X$

What do we mean by *outcome*?

Outcome Classes

Outcome class: set of games where the same player always has a winning strategy.

Who wins on 0? ("Next" or "Previous" player?) A: Previous

Let's call the set of these positions $\mathcal{P}$, "Zero".

Bonus: $\forall X \in \mathcal{P} : X = 0$

Outcome Classes

Who wins on $*$? A: Next player!

Let's call these positions $\mathcal{N}$, "Fuzzy".

What's true of all single Nim heaps with at least one stick?

They're all in fuzzy!

Not all fuzzy games are equal to some single value, so they're... "fuzzy".

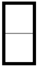
Fuzzy + Fuzzy

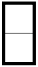
$$\square \in \mathcal{N}$$

What about $\square + \square$?

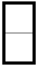
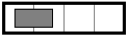
Sum $\in \mathcal{P}$, so it equals 0.

Fuzzy + Fuzzy

 $\in \mathcal{N}$,  $\in \mathcal{N}$

What about  $+$  ?

Still $\mathcal{N}$

Winning Move:  $+$ 

Outcome class defined by *existence* of a strategy. Not every move will be winning.

Different Fuzzy Values: Cram

$$= \{*, 0, *\} \;\; = \{0, *\}$$

Can win, but also has more power...

Let's call this $*2$.

Different Fuzzy Values: Nim

What's the value of the single Nim heap: [2]?

$$
\begin{aligned}
[2] &= \{\ [\ ],\ [1]\ \} \\
&= \{\ 0,\ \ *\ \} \\
&= *2
\end{aligned}
$$

What's the value of [3]?

$$
\begin{aligned}
[3] &= \{\ [\ ],\ [1],\ [2]\ \} \\
&= \{\ 0,\ \ *,\ *2\ \} \\
&= *3
\end{aligned}
$$

True for all single Nim heaps! $[k] = *k$

Nimbers!

More generally, $*k = \{*0, *1, *2, *3, \ldots, *(k-1)\}$

"$*k$ has all nimbers 0 through k-1, but not k"

Note: it doesn't matter whether there are nimbers above k.

"Minimal Excluded Value" (mex)

$\text{mex}(S \subset (\mathbb{N} \cup \{0\})) = \text{min}_{x \notin S}(\mathbb{N})$

Cram Nimbers

 $= \{\ \} = 0$

 $= \{0\} = *$

 $= \{0, 0\} = \{0\} = *$

 $= \{*, 0, *\} = *2$

 $= \{$    $, \} = \{ *, *, *, *\} = 0$

 $= \{*2, *, 0\} = *3$

 $= \{0, *2, 0\} = *$

 $= \{*3, 0, *3, 0\} = *$

Adding Nimbers

Bonus: summing nimbers is easy!

$*k + *m = \ *(k \oplus m)$

$$
\begin{aligned}
*6 + *5 &= *(110 \oplus 101) \\
& \quad\ \ 110 \\
& \underline{\oplus\ 101} \\
&= \quad 011 \\
&= \quad *3
\end{aligned}
$$

- $*8 + *7 = *15$
- $* + * = 0. \quad *k + *k = 0$
- $*10 + *13 = \ *7$

Cram Nimbers

 $= *$

 $= *$

 $= *2$

 $= \{*, *, *, *\} = 0$

 $= \{*2, *, 0\} = *3$

 $= \{0, *2, 0\} = *$

 $= \{*3, 0, *3, 0\} = *$

Impartial vs Partisan

So far both players have always had the same moves. ("Impartial games")

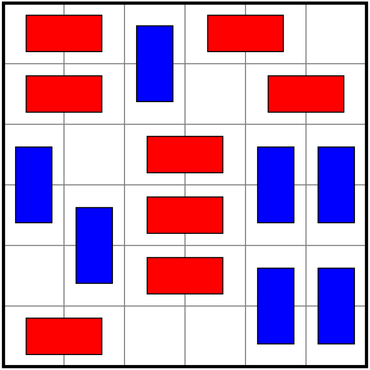
Nimbers cover all values for impartial games.

Different: "Partisan games". Player identity matters!

Left vs Right: Blue vs Red.

- New outcome classes!
- New representation!
- New values!

Domineering



Partisan Outcome Classes

Previously...

- $\mathcal{P}$: Prevous player wins, "Zero"
- $\mathcal{N}$: Next player wins, "Fuzzy"

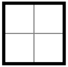
New possibilities:

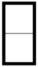
- $\mathcal{L}$: Left wins, "Positive"
- $\mathcal{R}$: Right wins, "Negative"

Notation: $o(G) =$ outcome class of G

Domineering Outcomes

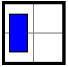
 $\in \mathcal{P}$

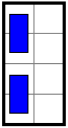
 $\in \mathcal{N}$

 $\in \mathcal{L}$

 $\in \mathcal{R}$

1 and 2

 $= 1$ "One move for Left"

 $= 2$ "Two moves for Left"

Partisan Representations

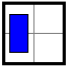
$$\{ \text{ Left's options } \mid \text{ Right's options } \}$$

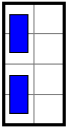
- $0 = \{ \mid \}$
- $* = \{0 \mid 0\}$
- $*2 = \{0, * \mid 0, *\}$
- etc...

What happens if the two sides aren't the same?

Partisan Representations

$$\{\ \text{Left's options} \mid \text{Right's options}\ \}$$

 $= 1 = \{0 \mid \}$

 $= 2 = \{1, 0 \mid \}$

We'd really like to drop that 0 inside of 2...

Intuition: $0 < 1$, so we can remove the 0 option, because Left will always choose 1 instead. How can we define $<$?

Comparing Games

$G < H$ if Left never prefers G to H

Never? In the context of other games, X

- $G \leq H$ if Left wins on $G + X \Rightarrow$ Left wins on $H + X$
- Recall: $G = H$ if $\forall X : o(G + X) = o(H + X)$
- So: $G < H$ if $G \leq H$ and $\exists X : o(G + X) \neq \mathcal{L} = o(H + X)$

  $0 < 1$ and $* < 1$

- $G \| H$ if $G \not\geq H$ and $G \not\leq H$

  $0 \| *$ and $0 \| *2$

Dominated Moves & Integers

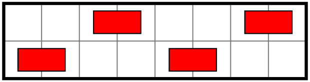
$\{0, 1 \mid \cdots\} = \{1 \mid \cdots\}$

$\{\cdots \mid *, 1\} = \{\cdots \mid *\}$

$\{1 \mid \} = 2$

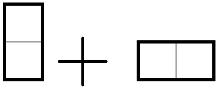
$\forall k \in \mathbb{N}:$

- $\{k \mid \} = k + 1$
- $\{ \mid -k\} = -k - 1$



Negatives

What's the outcome class of $1 + -1$?



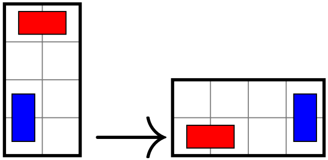
$1 + -1 \in \mathcal{P}$

In general,

$$
\begin{aligned}
-G &= -\{L_1, \ldots, L_n \mid R_1, \ldots, R_m\} \\
&= \{-R_1, \ldots, -R_m \mid -L_1, \ldots, -L_n\} \\
-(k+1) &= -\{k \mid \} \\
&= \{ \mid -k\} \\
&= -k-1
\end{aligned}
$$

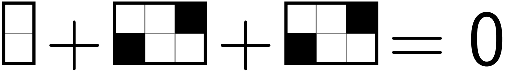
Domineering Negatives

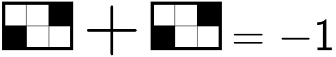
In Domineering, just reflect and switch colors!



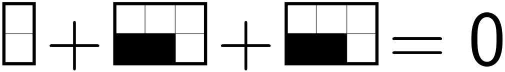
Domineering Halves

What's this sum?



So...

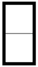
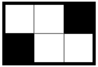
What about this?

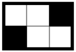


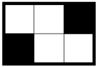
Same thing!

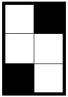
Domineering Halves

What about this?

 $+$  $+$  $\neq 0$

So,  $+$  $\neq -1$

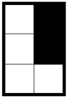
 $=$  ?

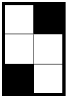
 $+$  $= 0$?

Only one of these is $-1/2$...

Domineering Halves

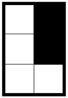
Look at the outcome classes!

 $\in \mathcal{L}$ $(= 1/2)$

 $\in \mathcal{N}$ $(= 1/2 + *)$

Two times either one is 1

Simplest Numbers

 $= \{0 \mid 1\} = 1/2$

Simplest numbers!

If $m < n$ (both numbers), then $\{m \mid n\} = k$, where:

- If there is any integer strictly between m and n, (i.e., $k \in (m, n)$) then k is the one with smallest absolute value.
- Otherwise,

$$k = \frac{a}{2^b}$$

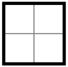
where $m < k < n$ and $b > 0$ is minimal.

Simplest Numbers

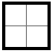
Examples!

- $\{0 \mid 5\} = 1$
- $\{-99 \mid 5\} = 0$
- $\{99 \mid 103\} = 100$
- $\{0 \mid 1\} = 1/2$
- $\{3/4 \mid 1\} = 7/8$

What about this?

 $= \{1 \mid -1\}$ No simplest number here!

Switches

 $= \{1 \mid -1\} \neq 0$

In general, $\pm k = \{k \mid -k\}$

- Not surreal
- "Hot" games
- $* = \pm 0$
- $\{6 \mid 0\} = 3 + \{3 \mid -3\} = 3 \pm 3$
- Sums possible! E.g. $2 \pm 6 \pm 3 \pm 2$

Infinitesimals

$\uparrow = \{0 \mid *\}$

- $\in \mathcal{L}$
- Smaller than any positive number!
- "Dicot" games
- Even smaller: Tinys and Minys!

So much more!

"Math":

- Which values from which rulesets?
- Misere Play, Scoring Games
- Other Sums

"Computer Science":

- Computational Complexity
- AI

Thank you!